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MATHEMATICS-II A

(ENGLISH MEDIUM)

BASIC LEARNING MATERIAL

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PREFACE

The ongoing Global Pandemic Covid-19 that has engulfed the entire world has changed every sphere of our life. Education, of course is not an exception. In the absence and disruption of Physical Classroom Teaching, Department of Intermediate Education Telangana has successfully engaged the students and imparted education through TV lessons. In the back drop of the unprecedented situation due to the pandemic TSBIE has reduced the burden of curriculum load by considering only 70% syllabus for class room instruction as well as for the forthcoming Intermediate Examinations. It has also increased the choice of questions in the examination pattern for the convenience of the students.

To cope up with exam fear and stress and to prepare the students for annual exams in such a short span of time , TSBIE has prepared "Basic Learning Material" that serves as a primer for the students to face the examinations confidently. It must be noted here that, the Learning Material is not comprehensive and can never substitute the Textbook. At most it gives guidance as to how the students should include the essential steps in their answers and build upon them. I wish you to utilize the Basic Learning Material after you have thoroughly gone through the Text Book so that it may enable you to reinforce the concepts that you have learnt from the Textbook and Teachers. I appreciate ERTW Team, Subject Experts, who have involved day in and out to come out with the Basic Learning Material in such a short span of time.

I would appreciate the feedback from all the stake holders for enriching the learning material and making it cent percent error free in all aspects.

The material can also be accessed through our website**www.tsbie.cgg.gov.in.**

Commissioner & Secretary
Intermediate Education, Telangana.

CONTENTS

Unit 1

Complex Numbers

A complex number is an ordered pair of real numbers(a, b). The set of all complex numbers is denoted by $C = \{(a, b) / a \in R, b \in R\} = R \times R$

Note :- i) $Z = (a, b) = a + ib$ where $i = \sqrt{-1}$ or $i^2 = -1$

ii) If $(a, b) = (c, d) \Rightarrow a = c; b = d$

iii) **Addition :-** If $Z_1=(a, b)$ and $Z_2=(c, d)$ then

$Z_1+Z_2 = (a + c, b + d)$

iv) If $Z = (a, b)$ then $-Z = (-a, -b)$

v) **Subtraction :-** If $Z_1=(a, b)$ and $Z_2=(c, d)$ then

$Z_1-Z_2 = (a - c, b - d)$

vi) **Multiplication :-**

If $Z_1=(a, b)$ and $Z_2=(c, d)$ then

$Z_1.Z_2 = (a, b).(c, d) = (ac - bd, ad + bc)$

vii) **Division :-** $\alpha = (a, b), \beta = (c, d)$ & $\beta \neq (0, 0)$ then

$$\frac{\alpha}{\beta} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$$

viii) If $\beta \neq (0, 0)$ and if $\beta = (a, b)$ then multiplicative inverse of β is $\beta^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

Conjugate of a Complex number :

For any Complex number $Z = a + ib$, we define the conjugate of Z is $\overline{Z} = a - bi$

Note: If $\alpha, \beta \in C$, then

(i) $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$, ii) $\overline{\alpha.\beta} = \overline{\alpha}.\overline{\beta}$, iii) $\overline{\overline{\alpha}} = \alpha$, iv) If $\beta \neq 0$, then $\left(\overline{\alpha/\beta}\right) = \overline{\alpha}/\overline{\beta}$

Square Root of a Complex number :

If $Z = a + ib$, the square root of 'Z' is defined as

$$Z^{1/2} \text{ or } \sqrt{Z} = \sqrt{a+ib} = \begin{cases} \pm\left[\sqrt{\dfrac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}\right], & \text{if } b > 0 \\ \pm\left[\sqrt{\dfrac{\sqrt{a^2+b^2}+a}{2}} - i\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}\right], & \text{if } b < 0 \end{cases}$$

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1. If $Z_1=(2, -1), Z_2=(6, 3)$, find $Z_1 - Z_2$

Sol. $Z_1=(2, -1), Z_2=(6, 3)$

$Z_1 - Z_2 = (2 - 6, -1 - 3) = (-4, -4)$.

2. Write the additive inverse of (– 6, 5) + (10, – 4)

Sol. $(-6, 5) + (10, -4) = (4, 1) = 4 + i$.

Additive inverse of $4 + i = -(4 + i) = -4 - i$.

3. If $Z=(\cos\theta, \sin\theta)$, then find $z - \dfrac{1}{z}$.

Sol. $z = (\cos\theta, \sin\theta) = \cos\theta + i\sin\theta$

$$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta$$

Now $z - \dfrac{1}{z} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$

$= 2i\sin\theta = (0, 2\sin\theta)$.

4. Write the multiplicative inverse of (3, 4).

Sol. Multiplicative inverse of (3, 4) is $\left(\dfrac{3}{3^2+4^2}, \dfrac{-4}{3^2+4^2}\right) = \left(\dfrac{3}{25}, \dfrac{-4}{25}\right)$.

5. Write the multiplicative inverse of (7, 24).

Sol. Multiplicative inverse of (7, 24) is $\left(\dfrac{7}{7^2+24^2}, \dfrac{-24}{7^2+24^2}\right) = \left(\dfrac{7}{625}, \dfrac{-24}{625}\right)$.

6. If $Z_1 = (3, 5)$ and $Z_2 = (2, 6)$ find $Z_1 . Z_2$.

Sol. (a, b).(c, d)=(ac-bd, ad+bc)

$Z_1.Z_2=(3, 5).(2, 6)=(6-30, 18+10)=(-24, 28)$

i.e., $(3+5i)(2+6i)=-24+28i$

7. If $Z_1=(6, 3)$, $Z_2=(2, -1)$ then find Z_1/Z_2

Sol. If $\alpha=(a, b)$, $\beta=(c, d)$ then $\frac{\alpha}{\beta}=\left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$

$Z_1=(6, 3)$, $Z_2=(2, -1)$

$$\frac{Z_1}{Z_2}=\left(\frac{12-3}{4+1}, \frac{6+6}{4+1}\right)=\left(\frac{9}{5}, \frac{12}{5}\right) \text{ or } \frac{9}{5}+\frac{12}{5}i$$

8. Write the complex number $(2-3i)(3+4i)$ in the form A + iB.

Sol. $(2-3i)(3+4i)=6+8i-9i+12=18-i=18+i(-1)$.

9. Write the complex number $3(7+7i)+i(7+7i)$ in the form A + iB.

Sol. $3(7+7i)+i(7+7i)=21+21i+7i-7=14+28i$.

10. Write the Complex number $\frac{4+3i}{(2+3i)(4-3i)}$ in the form of A+iB

Sol. $$\frac{4+3i}{(2+3i)(4-3i)}=\frac{4+3i}{8-6i+12i+9}$$

$$=\frac{4+3i}{17+6i}$$

$$=\frac{(4+3i)(17-6i)}{(17+6i)(17-6i)}=\frac{68-24i+51i+18}{289+36}$$

$$=\frac{86+27i}{325}=\frac{86}{325}+\frac{27}{325}i$$

11. Write the Complex number $\frac{2+5i}{3-2i}+\frac{2-5i}{3+2i}$ in the form of A+iB

Sol. $$\frac{2+5i}{3-2i}+\frac{2-5i}{3+2i}=\frac{(2+5i)(3+2i)}{(3-2i)(3+2i)}+\frac{(2-5i)(3-2i)}{(3+2i)(3-2i)}$$

$$=\frac{6+4i+15i-10}{9+4}+\frac{6-4i-15i-10}{9+4}$$

$$= \frac{-4+19i}{13} + \frac{-4-19i}{13}$$

$$= \frac{-4+19i-4-19i}{13}$$

$$= \frac{-8}{13} + i(0)$$

12. Write the Complex number i^{-19} in the form A+iB

Sol. $i^{-19} = \frac{1}{i^{19}}$

$$= \frac{1}{i^{18}.i}$$

$$= \frac{1}{\left(i^2\right)^9 i}$$

$$= \frac{1}{\left(-1\right)^9 i} \quad \left(\because i^2 = -1\right)$$

$$= \frac{1}{-i}$$

$$= \frac{1(i)}{-i(i)}$$

$$= i \quad \left(\because i^2 = -1\right)$$

$$= 0 + i.1$$

13. Write the conjugate of the Complex number (3 + 4i)

Sol. Conjugate of the Complex number (3 + 4i) is (3– 4i).

14. Write the conjugate of the Complex number $\frac{5i}{7+i}$

Sol. $\frac{5i}{7+i} = \frac{5i(7-i)}{(7+i)(7-i)} = \frac{35i-5i^2}{7^2-i^2}$

$$= \frac{35i+5}{50} = \frac{7i+1}{10} = \frac{1+7i}{10}.$$

It's Complex Conjugate number is $\frac{1-7i}{10}$.

15. Write the conjugate of the Complex number (2 + 5i)(-4 + 6i).

Sol. $(2+5i)(-4+6i) = -8+12i-20i+30 =$

$= 22-8i$. Its Complex Conjugate = $22 + 8i$.

16. Simplify $i^2 + i^4 + i^6 + ...(2n+1)$ terms

Sol. $i^2 + i^4 = -1 + (-1)^2 = 0$

Similarly $i^6 + i^8 = (i^2)^3 + (i^2)^4 = 0$

Sum of any two consecutive terms is zero.

$\therefore$ last term $= \left(i^2\right)^{2n+1} = \left(-1\right)^{2n+1} = -1$

$\therefore$ $i^2 + i^4 + i^6 + ... + (2n+1)$ terms = -1

17. Write the multiplicative inverse of i^{-35}

Sol. $i^{-35} = \dfrac{1}{i^{35}}$

$= \dfrac{1}{\left(i^2\right)^{17} i}$

$= \dfrac{1}{-i}$

$= \dfrac{1.i}{-i.i} = i$

Multiplicative inverse of $a + ib = \dfrac{a-ib}{a^2+b^2}$

$\therefore$ Mulitplicative inverse of 'i' $= \dfrac{-i}{(1)^2} = -i$

Problems for Practice

(i) Write the Complex number i^9 in the form of $A+iB$ Ans: $0+i.1$

(ii) Write the Complex number $(-i)(2i)$ in the form of $A+iB$ Ans: $2+i.0$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

1. Find the square root of 7 + 24i.

Sol:- Let $\sqrt{7+24i} = \pm(x+iy)$

$y = 24 > 0$

$$\therefore \sqrt{x+iy} = \pm\left[\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right]$$

$$\sqrt{7+24i} = \pm\left[\sqrt{\frac{\sqrt{7^2+24^2}+7}{2}} + i\sqrt{\frac{\sqrt{7^2+24^2}-7}{2}}\right]$$

$$= \pm\left(\sqrt{\frac{32}{2}} + i\sqrt{\frac{18}{2}}\right) = \pm(4+3i)$$

2. Show that $\mathbf{\frac{2-i}{(1-2i)^2}}$ and $\mathbf{\frac{-2-11i}{25}}$ are conjugate to each other.

Sol:- $\frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i}$

$$= \frac{2-i}{-3-4i}$$

$$= \frac{(2-i)(-3+4i)}{(-3-4i)(-3+4i)} = \frac{-6+8i+3i-4i^2}{9+16}$$

$$= \frac{-2+11i}{25} = \frac{-2}{25} + \frac{11i}{25}$$

Its Conjugate is $\frac{-2}{25} - \frac{11i}{25} = \frac{-2-11i}{25}$

$\therefore \frac{2-i}{(1-2i)^2}$ and $\frac{-2-11i}{25}$ are Conjugate to each other.

3. Find the least positive integer 'n', satisfying $\mathbf{\left(\frac{1+i}{1-i}\right)^n = 1}$

Sol. $\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)}$

$$= \frac{(1+i)^2}{(1+1)}$$

$$= \frac{1+i^2+2i}{2}$$

$$= \frac{1-1+2i}{2}$$

$$= \frac{2i}{2} = i$$

$\therefore$ The least positive integer 'n' such that $i^n = 1$ is 4

4. If $\mathbf{x + iy = \dfrac{1}{1 + Cos\,\theta + i\,Sin\,\theta}}$ then, Show that $\mathbf{4x^2 - 1 = 0}$.

Sol. $x + iy = \dfrac{1}{1 + Cos\theta + i\,Sin\theta}$

$$= \frac{1}{2Cos^2\frac{\theta}{2} + i(2)Sin\frac{\theta}{2}.Cos\frac{\theta}{2}}$$

$$= \frac{1}{2Cos\frac{\theta}{2}\left[Cos\frac{\theta}{2} + i\,Sin\frac{\theta}{2}\right]}$$

$$= \frac{Cos\frac{\theta}{2} - i\,Sin\frac{\theta}{2}}{2Cos\frac{\theta}{2}\left(Cos\frac{\theta}{2} + i\,Sin\frac{\theta}{2}\right)\left(Cos\frac{\theta}{2} - i\,Sin\frac{\theta}{2}\right)}$$

$$= \frac{Cos\frac{\theta}{2} - i\,Sin\frac{\theta}{2}}{2Cos\frac{\theta}{2}\left(Cos^2\frac{\theta}{2} + Sin^2\frac{\theta}{2}\right)}$$

$$= \frac{Cos\frac{\theta}{2} - i\,Sin\frac{\theta}{2}}{2Cos\frac{\theta}{2}}$$

$$x + iy = \frac{1}{2} - \frac{i\,Tan\frac{\theta}{2}}{2}$$

Equating real parts $x = \dfrac{1}{2}$

$\Rightarrow 2x = 1$, Squaring on both sides

$\Rightarrow 4x^2 = 1$

$\Rightarrow 4x^2 - 1 = 0$

5. If $x+iy=\dfrac{3}{2+\cos\theta+i\sin\theta}$ then, Show that $x^2+y^2=4x-3$

Sol. $x+iy=\dfrac{3}{2+Cos\theta+i\,Sin\theta}$

$$=\frac{3\left[(2+Cos\theta)-i\,Sin\theta\right]}{\left[(2+Cos\theta)+i\,Sin\theta\right]\left[(2+Cos\theta)-i\,Sin\theta\right]}$$

$$=\frac{(6+3Cos\theta)-3i\,Sin\theta}{(2+Cos\theta)^2+Sin^2\theta}$$

$$=\frac{6+3Cos\theta-3i\,Sin\theta}{4+4Cos\theta+(Sin^2\theta+Cos^2\theta)}$$

$$=\frac{6+3Cos\theta-3i\,Sin\theta}{5+4Cos\theta}$$

$$x+iy=\frac{6+3Cos\theta}{5+4Cos\theta}-\frac{i(3\,Sin\theta)}{5+4Cos\theta}$$

Equating real and imaginary parts $x=\dfrac{6+3Cos\theta}{5+4Cos\theta},\ y=\dfrac{-3\,Sin\theta}{5+4Cos\theta}$

L.H.S:- x^2+y^2

$$=\left(\frac{6+3Cos\theta}{5+4Cos\theta}\right)^2+\left(\frac{-3\,Sin\theta}{5+4Cos\theta}\right)^2$$

$$=\frac{36+9Cos^2\theta+36Cos\theta}{(5+4Cos\theta)^2}+\frac{9Sin^2\theta}{(5+4Cos\theta)^2}$$

$$=\frac{36+9Cos^2\theta+9Sin^2\theta+36Cos\theta}{(5+4Cos\theta)^2}$$

$$=\frac{36+36Cos\theta+9\left(Cos^2\theta+Sin^2\theta\right)}{(5+4Cos\theta)^2}$$

$$=\frac{45+36Cos\theta}{(5+4Cos\theta)^2}$$

$$=\frac{9(5+4Cos\theta)}{(5+4Cos\theta)^2}$$

$$=\frac{9}{(5+4Cos\theta)}$$

R.H.S:- $4x-3$

$= 4\left(\frac{6+3Cos\theta}{5+4Cos\theta}\right) - 3$

$= \frac{24+12\,Cos\theta - 15 - 12\,Cos\theta}{(5+4Cos\theta)}$

$= \frac{9}{(5+4Cos\theta)}$

$\therefore L.H.S. = R.H.S.$

6. If z = 3 – 5i, then show that $z^3 - 10z^2 + 58z - 136 = 0$.

Sol. $z = 3 - 5i \Rightarrow z - 3 = -5i$

$\Rightarrow (z-3)^2 = (-5i)^2$

$\Rightarrow z^2 - 6z + 9 = -25$

$\Rightarrow z^2 - 6z + 34 = 0.$

$z^3 - 10z^2 + 58z - 136 = z(z^2 - 6z + 34) - 4(z^2 - 6z + 34)$

$= z(0) - 4(0) = 0.$

7. Find the real values of θ in order that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is a

(a) real number (b) purely imaginary number

Sol:- $\frac{3+2i\sin\Theta}{1-2i\sin\Theta}$

$= \frac{(3+2i\sin\Theta)(1+2i\sin\Theta)}{(1-2i\sin\Theta)(1+2i\sin\Theta)}$

$= \frac{3+4i^2\sin^2\Theta + 8i\sin\Theta}{(1+4\sin^2\Theta)}$

$= \frac{3-4\sin^2\Theta}{1+4\sin^2\Theta} + \frac{8\sin\Theta}{1+4\sin^2\Theta}i$

a) If the given expression is purely real then $\frac{8\sin\Theta}{1+4\sin^2\Theta} = 0$

$\Rightarrow \sin\Theta = 0 \Rightarrow \Theta = n\pi, n \in Z$

b) If the given expression is purely imaginary then $= \frac{3-4\sin^2\Theta}{1+4\sin^2\Theta} = 0$

$\Rightarrow 3-4\sin^2\Theta=0$

$\Rightarrow \sin^2\Theta=\frac{3}{4}$

$\Rightarrow \sin^2\Theta=\left(\frac{\sqrt{3}}{2}\right)^2=\sin^2\left(\frac{\pi}{3}\right)$

$\Rightarrow \theta=2n\pi\pm\frac{\pi}{3},\ n\in Z$

Problems for Practice

(i) Find the Square root of -8-6i

Ans: $\pm(1-3i)$

Hint : Here b=-6<0, if b<0, then $\sqrt{a+ib}=\pm\left(\sqrt{\frac{r+a}{2}}-i\sqrt{\frac{r-a}{2}}\right)$

(ii) Find the square root of (-5+12i)

Ans: $\pm(2+3i)$

Hint : Refer Ex.3 from page 14 of text book

(iii) Show that $Z_1=\frac{2+11i}{25}$, $Z_2=\frac{-2+i}{(1-2i)^2}$ are Conjugate to each other.

Hint : Refer Example '2' from page No. 13 of text book.

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Unit 2

De Moivre's Theorem

1) **De moivre's Theorem for integral index :-** For any real number 'θ' and any integer 'n',

$(\text{Cos}\theta + i\,\text{Sin}\theta)^n = \text{Cos}(n\theta) + i\,\text{Sin}(n\theta)$

2) $(\text{Cos}\theta + i\,\text{Sin}\theta)$ is also written as '$\text{Cis}\theta$'

3) $(\text{Cos}\theta + i\,\text{Sin}\theta)^{-n} = \text{Cos}\,n\theta - i\,\text{Sin}\,n\theta$, Where 'n' is an integer.

4) $\cos\theta + i\sin\theta = \dfrac{1}{\cos\theta - i\sin\theta}$ and $\cos\theta - i\sin\theta = \dfrac{1}{\cos\theta + i\sin\theta}$

5) $(\text{Cos}\theta - i\,\text{Sin}\theta)^n = \left(\dfrac{1}{\text{Cos}\theta + i\,\text{Sin}\theta}\right)^n = (\text{Cos}\theta + i\,\text{Sin}\theta)^{-n} = \text{Cos}\,n\theta - i\,\text{Sin}\,n\theta$ Where n is an integer.

6) $\text{Cis}\theta.\text{Cis}\phi = \text{Cis}(\theta + \phi)$ for any $\theta, \phi \in R$

7) Cube roots of unity are $1, \omega, \omega^2$ Where $\omega = \dfrac{-1 + i\sqrt{3}}{2}$ and $\omega^2 = \dfrac{-1 - i\sqrt{3}}{2}$

8) $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1. If A, B, C are the angles of a triangle and $x = cis\,A, y = cis\,B, z = cis\,C$, find the value of *xyz*.

Sol:- $xyz = cis\,A\,cis\,B\,cis\,C = cis(A+B+C) = \cos\pi + i\sin\pi = -1.$

$\because xyz = -1$

2. If $x = \cos\theta + i\sin\theta$, find $x^6 + \dfrac{1}{x^6}$.

Sol:- $x^6 = (\cos\Theta + i\sin\Theta)^6 = \cos 6\Theta + i\sin 6\Theta$ and $\dfrac{1}{x^6} = \cos 6\Theta - i\sin 6\Theta.$

$\therefore x^6 + \dfrac{1}{x^6} = 2\cos 6\Theta.$

3. **If $1,\omega,\omega^2$ are the cube roots of unity, then find the value of $(1-\omega+\omega^2)^3$.**

Sol:- $(1-\omega+\omega^2)^3 = (1+\omega^2-\omega)^3$ $\left[\begin{array}{l}\because 1+\omega+\omega^2=0 \\ \Rightarrow 1+\omega^2=-\omega,\ \omega^3=1\end{array}\right]$

$$= (-\omega-\omega)^3 = (-2\omega)^3 = -8\omega^3 = -8$$

4. **If $1,\omega,\omega^2$ are the cube roots of unity, then fine the value of $(1+\omega)^3+(1+\omega^2)^3$**

Sol:- $(1+\omega)^3+(1+\omega^2)^3 = (-\omega^2)^3+(-\omega)^3 = -(\omega^3)^2-\omega^3 = -1-1=-2$.

5. **Find the cube roots of '8' ?**

Sol:- $8^{\frac{1}{3}} = \left[(8)(1)\right]^{\frac{1}{3}} = 8^{\frac{1}{3}}(1)^{\frac{1}{3}} = \left(2^3\right)^{\frac{1}{3}}(1)^{\frac{1}{3}}$

$= 2(1)^{\frac{1}{3}}$

$= 2(1),\ 2(\omega), 2(\omega^2)$

$\therefore$ Cube roots of '8' are $2,\ 2\omega,\ 2\omega^2$

6. **Find the value of $\left(1+i\sqrt{3}\right)^3$**

Sol:- $1+i\sqrt{3} = 2\left[\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right] = 2\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$

$$\left(1+i\sqrt{3}\right)^3 = \left[2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^3 = 2^3\left[\cos\pi+i\sin\pi\right] = 8[-1+i(0)] = -8$$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

1. **Simplify $\dfrac{(\mathbf{Cos}\,\alpha+\mathbf{i}\,\mathbf{Sin}\,\alpha)^4}{(\mathbf{Sin}\,\beta+\mathbf{i}\,\mathbf{Cos}\,\beta)^8}$**

Sol:- $\dfrac{(Cos\,\alpha+i\,Sin\,\alpha)^4}{(Sin\,\beta+i\,Cos\,\beta)^8} = \dfrac{(Cos\,\alpha+i\,Sin\,\alpha)^4}{(-i^2 Sin\,\beta+i\,Cos\,\beta)^8}$

$$= \frac{(Cos\,\alpha+i\,Sin\,\alpha)^4}{\left[i(Cos\,\beta-i\,Sin\,\beta)\right]^8}$$

$$= \frac{(Cos\,\alpha+i\,Sin\,\alpha)^4}{(i)^8(Cos\,\beta-i\,Sin\,\beta)^8}$$

$$= (Cos\,\alpha+i\,Sin\,\alpha)^4(Cos\,\beta-i\,Sin\,\beta)^{-8} \quad \left(\because i^8=\left(i^2\right)^4=(-1)^4=1\right)$$

$$= (Cos\,4\alpha+i\,Sin\,4\alpha)(Cos\,8\beta+i\,Sin\,8\beta)$$

$= Cos(4\alpha + 8\beta) + i\,Sin(4\alpha + 8\beta)$ or

$= Cis(4\alpha + 8\beta)$

2. Find the value of $(1 - i)^8$

Sol:- Let $a + ib = 1 - 1i$

$\Rightarrow a = 1,\ b = -1,\ \sqrt{a^2 + b^2} = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

$(1 - i)^8 = \left[\sqrt{2}\left(\frac{1-i}{\sqrt{2}}\right)\right]^8$

$= \left(\sqrt{2}\right)^8 \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^8$

$= \left(\sqrt{2}\right)^8 \left(Cos\,45^0 - i\,Sin\,45^0\right)^8$

$= 2^4\left[Cos(8 \times 45) - i\,Sin(8 \times 45)\right] = 16\left[Cos\,(360^0) - i\,Sin(360^0)\right] = 16\left[1 - i(0)\right] = 16$

3. Find the value of $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$

Sol:- $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = \left(Cos\,30^0 + i\,Sin\,30^0\right)^5 - \left(Cos\,30^0 - i\,Sin\,30^0\right)^5$

$= \left(Cos\,150^0 + i\,Sin\,150^0\right) - \left(Cos\,150^0 - i\,Sin\,150^0\right)$

$= Cos\,150^0 + i\,Sin\,150^0 - Cos\,150^0 + i\,Sin\,150^0$

$= 2i\,Sin\,150^0$ $\left[\therefore Sin150^0 = Sin\left(180^0 - 30^0\right) = Sin30^0 = \frac{1}{2}\right]$

$= 2i\left(\frac{1}{2}\right) = i$

4. If $1, \omega, \omega^2$ are the cube roots of unity, then find the value of

$\left(1 - \omega + \omega^2\right)^5 + \left(1 + \omega - \omega^2\right)^5$

Sol:- If $1, \omega, \omega^2$ are the cube roots of unity, then $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

$\left(1 - \omega + \omega^2\right)^5 + \left(1 + \omega - \omega^2\right)^5 = \left[\left(1 + \omega^2\right) - \omega\right]^5 + \left[\left(1 + \omega\right) - \omega^2\right]^5$

$= \left[-\omega - \omega\right]^5 + \left[-\omega^2 - \omega^2\right]^5$

$= \left[-2\omega\right]^5 + \left[-2\omega^2\right]^5$

$= (-2)^5\left[\omega^5 + \omega^{10}\right]$

$= -32\left[\omega^3.\omega^2 + \left(\omega^3\right)^3.\ \omega\right]$

$$= -32\left[\omega^2 + \omega\right] \qquad \left[\begin{array}{l}\because \omega^3 = 1 \\ 1+\omega+\omega^2 = 0\end{array}\right]$$

$$= -32\left[-1\right]$$

$$= 32$$

$\therefore$ The value of $\left(1-\omega+\omega^2\right)^5 + \left(1+\omega-\omega^2\right)^5$ is 32

5. **If $1, \omega, \omega^2$ are the cube roots of unity, then prove that**

i) $\dfrac{1}{1+2\omega} + \dfrac{1}{2+\omega} = \dfrac{1}{1+\omega}$

ii) $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$

Sol:- i) $\dfrac{1}{1+2\omega} + \dfrac{1}{2+\omega} = \dfrac{1}{1+\omega} \Rightarrow \dfrac{1}{1+2\omega} + \dfrac{1}{2+\omega} - \dfrac{1}{1+\omega} = 0$

L.H.S. = $\dfrac{1}{1+2\omega} + \dfrac{1}{2+\omega} - \dfrac{1}{1+\omega} = \dfrac{1+2\omega+2+\omega}{(1+2\omega)(2+\omega)} - \dfrac{1}{(1+\omega)}$

$$= \frac{3+3\omega}{2+4\omega+\omega+2\omega^2} - \frac{1}{-\omega^2} = \frac{3(1+\omega)}{2+2\omega+2\omega^2+3\omega} + \frac{1}{\omega^2}$$

$$= \frac{3(1+\omega)}{2(1+\omega+\omega^2)+3\omega} + \frac{\omega^3}{\omega^2}$$

$$= \frac{3(1+\omega)}{3\omega} + \omega = \frac{-\omega^2}{\omega} + \omega = -\omega + \omega = 0 = \text{RHS.} \qquad \left[\begin{array}{r}\because\ 1+\omega+\omega^2 = 0 \\ \omega^3 = 1\end{array}\right]$$

ii) $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$$= (2-\omega)^2(2-\omega^2)^2 = \left[(2-\omega)(2-\omega^2)\right]^2 = [4-2\omega^2-2\omega+\omega^3]^2 \qquad \left[\because\ \omega^3 = 1\right]$$

$$= [5-2(\omega^2+\omega)]^2 = [5-2(-1)]^2 = (5+2)^2 = 49$$

6. **If $1, \omega, \omega^2$ are the cube roots of unity, then find the value of**

$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$ $\qquad \left[\begin{array}{l}\because \omega^3 = 1 \\ 1+\omega+\omega^2 = 0 \\ 1 = -\omega-\omega^2\end{array}\right]$

Sol: $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8) = (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$$= \left[(1-\omega)(1-\omega^2)\right]^2 = (1-\omega-\omega^2+\omega^3)^2 = (2+1)^2 = 9.$$

LONG ANSWER TYPE QUESTIONS (7 MARKS)

1. **If 'n' is an integer show that $(1+i)^{2n} + (1-i)^{2n} = 2^{n+1}\text{Cos}\left(\dfrac{n\pi}{2}\right)$.**

Sol:- Let $1+i = a+ib \Rightarrow a = 1, b = 1$ and $\sqrt{a^2+b^2} = \sqrt{1+1} = \sqrt{2}$

$$(1+i)^{2n} = \left(\sqrt{2}\right)^{2n}\left(\frac{1+i}{\sqrt{2}}\right)^{2n}$$

$$= 2^{\frac{2n}{2}}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^{2n}$$

$$= 2^{n}\left(Cos\frac{\pi}{4}+i\,Sin\frac{\pi}{4}\right)^{2n}$$

$$= 2^{n}\left(Cos\frac{2n\pi}{4}+i\,Sin\frac{2n\pi}{4}\right)$$

$$\therefore (1+i)^{2n} = 2^{n}\left(Cos\frac{n\pi}{2}+i\,Sin\frac{n\pi}{2}\right) \text{..............................(i)}$$

Let $1-i = x+iy \Rightarrow x=1,\ y=-1$ and $\sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

$$(1-i)^{2n} = \left(\sqrt{2}\right)^{2n}\left(\frac{1-i}{\sqrt{2}}\right)^{2n}$$

$$= 2^{\frac{2n}{2}}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)^{2n}$$

$$= 2^{n}\left(Cos\frac{\pi}{4}-i\,Sin\frac{\pi}{4}\right)^{2n}$$

$$= 2^{n}\left(Cos\frac{2n\pi}{4}-i\,Sin\frac{2n\pi}{4}\right)$$

$$= 2^{n}\left(Cos\frac{n\pi}{2}-i\,Sin\frac{n\pi}{2}\right) \text{..............................(ii)}$$

Adding (i) & (ii)

$$(1+i)^{2n}+(1-i)^{2n} = 2^{n}\left(Cos\frac{n\pi}{2}+i\,Sin\frac{n\pi}{2}\right)+2^{n}\left(Cos\frac{n\pi}{2}-i\,Sin\frac{n\pi}{2}\right)$$

$$= 2^{n}\left[Cos\frac{n\pi}{2}+i\,Sin\frac{n\pi}{2}+Cos\frac{n\pi}{2}-i\,Sin\frac{n\pi}{2}\right]$$

$$= 2^{n}\left[2.\ Cos\frac{n\pi}{2}\right]$$

$$= 2^{n+1}Cos\frac{n\pi}{2} = R.H.S \quad \text{Hence Proved.}$$

2. If 'n' is a positive integer show that $(1+i)^n+(1-i)^n = 2^{\frac{n+2}{2}}Cos\left(\frac{n\pi}{4}\right)$.

Sol:- Let $1+i = a+ib \Rightarrow a=1,\ b=1$ and $\sqrt{a^2+b^2} = \sqrt{1+1} = \sqrt{2}$

Now, $(1+i)^n = \left(\sqrt{2}\right)^n \left(\frac{1+i}{\sqrt{2}}\right)^n$

$$= \left(\sqrt{2}\right)^n \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^n$$

$$= \left(\sqrt{2}\right)^n \left(Cos\frac{\pi}{4} + i\,Sin\frac{\pi}{4}\right)^n$$

$\therefore (1+i)^n = 2^{\frac{n}{2}}\left(Cos\frac{n\pi}{4} + i\,Sin\frac{n\pi}{4}\right)$ ....(i)

Let $1-i = x+iy \Rightarrow x=1,\ y=-1$ and $\sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

Now $(1-i)^n = \left(\sqrt{2}\right)^n \left(\frac{1-i}{\sqrt{2}}\right)^n$

$$= 2^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)^n$$

$$= 2^{\frac{n}{2}}\left(Cos\frac{\pi}{4} - i\,Sin\frac{\pi}{4}\right)^n$$

$\therefore (1-i)^n = 2^{\frac{n}{2}}\left(Cos\frac{n\pi}{4} - i\,Sin\frac{n\pi}{4}\right)$ ....(ii)

Adding (i) & (ii)

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}}\left(Cos\frac{n\pi}{4} + i\,Sin\frac{n\pi}{4}\right) + 2^{\frac{n}{2}}\left(Cos\frac{n\pi}{4} - i\,Sin\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}}\left(Cos\frac{n\pi}{4} + i\,Sin\frac{n\pi}{4} + Cos\frac{n\pi}{4} - i\,Sin\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}}\left(2.Cos\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}+1}\,Cos\frac{n\pi}{4}$$

$$= 2^{\frac{n+2}{2}}\,Cos\frac{n\pi}{4} = R.H.S.$$ Hence Proved

3. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$ then for any $n \in \mathrm{N}$, show that $\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$.

Sol:- $x^2 - 2x + 4 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}\,i$

Let $\alpha = 1+\sqrt{3}i, \beta = 1-\sqrt{3}i$

$$\alpha^n+\beta^n=(1+\sqrt{3}i)^n+(1-\sqrt{3}i)^n=\left[2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right]^n+\left[2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right]^n$$

$$=2^n\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n+2^n\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n$$

$$=2^n\left[\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}+\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right]=2^n.2\cos\frac{n\pi}{3}=2^{n+1}\cos\frac{n\pi}{3}.$$

4. If 'n' is an integer and $\mathbf{Z=Cis\,\theta,\left(\theta\neq(2n+1)\frac{\pi}{2}\right)}$, then show that $\mathbf{\frac{Z^{2n}-1}{Z^{2n}+1}=i\ Tan(n\theta)}$

Sol:- Given $Z=Cis(\theta)=Cos\theta+i\,Sin\theta$

L.H.S:- $\frac{Z^{2n}-1}{Z^{2n}+1}$

$$=\frac{(Cos\theta+i\,Sin\theta)^{2n}-1}{(Cos\theta+i\,Sin\theta)^{2n}+1}$$

$$=\frac{Cos\,2n\theta+i\,Sin\,2n\theta-1}{Cos\,2n\theta+i\,Sin\,2n\theta+1}$$

$$=\frac{-[1-Cos\,2n\theta]+i\,Sin\,2n\theta}{[1+Cos\,2n\theta]+i\,Sin\,2n\theta}$$

$$=\frac{-2\,Sin^2n\theta+i(2)Sin\,n\theta.Cos\,n\theta}{2\,Cos^2n\theta+i(2)\,Sin\,n\theta.Cos\,n\theta}$$

$$=\frac{2i^2Sin^2n\theta+2i\,Sin\,n\theta.Cos\,n\theta}{2\,Cos^2n\theta+2i\,Sin\,n\theta.Cos\,n\theta}\qquad(\because i^2=-1)$$

$$=\frac{2i\,Sin\,n\theta\,[Cos\,n\theta+i\,Sin\,n\theta]}{2\,Cos\,n\theta\,[Cos\,n\theta+i\,Sin\,n\theta]}$$

$=i\,Tan(n\theta)=R.H.S.$ Hence Proved.

5. If 'n' is an integer than show that

$$\mathbf{(1+Cos\theta+i\,Sin\theta)^n+(1+Cos\theta-i\,Sin\theta)^n=2^{n+1}Cos^n\left(\frac{\theta}{2}\right)Cos\left(\frac{n\theta}{2}\right)}$$

Sol:- L.H.S:- $(1+Cos\theta+i\,Sin\theta)^n+(1+Cos\theta-i\,Sin\theta)^n$

$$=\left[2\,Cos^2\left(\frac{\theta}{2}\right)+i(2)\,Sin\left(\frac{\theta}{2}\right)Cos\left(\frac{\theta}{2}\right)\right]^n+\left[2\,Cos^2\left(\frac{\theta}{2}\right)-i(2)\,Sin\left(\frac{\theta}{2}\right)Cos\left(\frac{\theta}{2}\right)\right]^n$$

$$=2^n\,Cos^n\left(\frac{\theta}{2}\right)\left[Cos\left(\frac{\theta}{2}\right)+i\,Sin\left(\frac{\theta}{2}\right)\right]^n+2^n\,Cos^n\left(\frac{\theta}{2}\right)\left[Cos\left(\frac{\theta}{2}\right)-i\,Sin\left(\frac{\theta}{2}\right)\right]^n$$

$$= 2^n Cos^n\left(\frac{\theta}{2}\right)\left[\left(Cos\left(\frac{\theta}{2}\right)+i\,Sin\left(\frac{\theta}{2}\right)\right)^n+\left(Cos\left(\frac{\theta}{2}\right)-i\,Sin\left(\frac{\theta}{2}\right)\right)^n\right]$$

$$= 2^n Cos^n\left(\frac{\theta}{2}\right)\left[Cos\left(\frac{n\theta}{2}\right)+i\,Sin\left(\frac{n\theta}{2}\right)+Cos\left(\frac{n\theta}{2}\right)-i\,Sin\left(\frac{n\theta}{2}\right)\right]$$

$$= 2^n Cos^n\left(\frac{\theta}{2}\right)\left[2\,Cos\left(\frac{n\theta}{2}\right)\right]$$

$$= 2^{n+1} Cos^n\left(\frac{\theta}{2}\right).Cos\left(\frac{n\theta}{2}\right) = R.H.S.\ \text{Hence Proved.}$$

6. If 1, ω, ω² are the cube roots of unity, prove that

$$\mathbf{(1-\omega+\omega^2)^6+(1-\omega^2+\omega)^6=128=(1-\omega+\omega^2)^7+(1+\omega-\omega^2)^7}$$

Sol:- Since $1,\ \omega,\ \omega^2$ are the cube roots of unity, $1+\omega+\omega^2=0 \Rightarrow 1+\omega^2=-\omega$ and $1+\omega=-\omega^2$

$$(1-\omega+\omega^2)^6+(1-\omega^2+\omega)^6=(-\omega-\omega)^6+(-\omega^2-\omega^2)^6=2^6(\omega^6+\omega^{12}) \qquad [\because \omega^3=1]$$

$$=2^6(2)=128 \qquad ....(1)$$

$$(1-\omega+\omega^2)^7+(1+\omega-\omega^2)^7=(-\omega-\omega)^7+(-\omega^2-\omega^2)^7$$

$$=(-2)^7(\omega^7+\omega^{14})=(-2)^7(\omega+\omega^2).$$

$$=(-128)(-1)=128 \qquad ....(2)$$

From (1) and (2)

$$(1-\omega+\omega^2)^6+(1-\omega^2+\omega)^6=128=(1-\omega+\omega^2)^7+(1+\omega-\omega^2)^7$$

7. If 1, ω, ω² are the cube roots of unity, then prove that

$$\mathbf{(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)=x^3+y^3+z^3-3xyz}.$$

Sol:- Given that $1,\ \omega,\ \omega^2$ are the cube roots of unity

$\therefore 1+\omega+\omega^2=0 \Rightarrow \omega+\omega^2=-1\ and\ \omega^3=1 \Rightarrow \omega^4=\omega^3.\omega=1\omega=\omega$

Consider $(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$

$$=x^2+xy\omega^2+xz\omega+xy\omega+y^2\omega^3+yz\omega^2+xz\omega^2+yz\omega^4+z^2\omega^3$$

$$=x^2+(xy+yz+zx)\omega+(xy+yz+zx)\omega^2+y^2+z^2$$

$$=x^2+y^2+z^2+(xy+yz+zx)(\omega+\omega^2)$$

$$=x^2+y^2+z^2-xy-yz-zx \quad (\therefore \omega+\omega^2=-1)$$

L.H.S :- $(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$

$$=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

$$=x^3+y^3+z^3-3xyz=R.H.S.$$

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Unit 3

Quadratic Expressions

- $ax^2 + bx + c$ is a Quadratic expression.

  Ex: $3x^2+2x+7,\ 3x^2-7$ are quadratic expressions.
- $ax^2 + bx + c = 0$ is a Quadratic equation.

  Ex: $3x^2+2x-5=0,\ 3x^2+2 = x+7$ are quadratic equations.
- The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$
- Discriminant : $\Delta = b^2 - 4ac$

Nature of the roots of a quadratic equation

Let α, β be the roots of the quadratic equation $ax^2+bx+c=0,$, where a, b, c are real numbers.

Case (i) :- $\Delta = 0 \Leftrightarrow \alpha = \beta = \dfrac{-b}{2a}$ (a repeated root or double root of $ax^2+bx+c=0$)

Case (ii) :- $\Delta > 0 \Leftrightarrow \alpha$ and β are real and distinct

Case (iii) :- $\Delta < 0 \Leftrightarrow \alpha$ and β are non real conjugate complex numbers.

Let a, b and c are rational numbers, α and β are the roots of the equation $ax^2+bx+c=0$. Then

(i) α, β are equal rational numbers if $\Delta = 0$

(ii) α, β are distinct rational numbers if Δ is the square of a non-zero rational number

(iii) α, β are conjugate surds of $\Delta > 0$ and Δ is not the square of a rational number.

Relation between coefficients and roots of a quadratic equation

Let α and β be the roots of the quadratic equation $ax^2+bx+c=0$

Sum of the roots :

$$\alpha + \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = \frac{-b}{a}$$

Product of the roots :

$$\alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{(-b)^2-(b^2-4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

➩ If α, β are the roots of $ax^2 + bx + c = 0$, then the equation can be written as

$a\left(x^2 - (\alpha+\beta)x + \alpha\beta\right) = 0$

i.e., $a(x-\alpha)(x-\beta) = 0$

Common root :

A necessary and sufficient condition for the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ to have a common root is $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$

Some Properties of quadratic equations :

Let $f(x) = ax^2 + bx + c = 0$ be a quadratic equation and α, β are its roots. Then

(i) If $c \neq 0$ then $\alpha\beta \neq 0$ and $f\left(\frac{1}{x}\right) = 0$ is an equation whose roots and $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

(ii) $f(x-k) = 0$ is an equation whose roots are $\alpha + k$ and $\beta + k$

(iii) $f(-x) = 0$ is an equation whose roots are $-\alpha$ and $-\beta$

(iv) $f(\frac{x}{k}) = 0$ is an equation whose roots are $k\alpha$ and $k\beta$, where $k \neq 0$.

Sign of quadratic expressions

If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$ with $\alpha < \beta$, then

(i) for $\alpha < x < \beta$, $ax^2 + bx + c$ and 'a' have opposite signs

(ii) for $x < \alpha$ or $x > \beta$, $ax^2 + bx + c$ and 'a' have the same signs.

(iii) Let $a, b, c \in R$ and $a \neq 0$. Then the roots of $ax^2 + bx + c = 0$ are non-real complex numbers if and only if $ax^2 + bx + c$ and 'a' have the same sign for all $x \in R$

(iv) Let $a, b, c \in R$ and $a \neq 0$. If the equation $ax^2 + bx + c = 0$ has equal roots, then $ax^2 + bx + c$ and 'a' have the same sign for all real 'x', except for $x = \frac{-b}{2a}$

Maximum and minimum values

If $a, b, c \in R$ and $a < 0$, then the expression $ax^2 + bx + c$ has maximum at $x = \frac{-b}{2a}$ and the maximum value is $\frac{4ac - b^2}{4a}$.

If $a, b, c \in R$ and $a > 0$, then the expression $ax^2 + bx + c$ has minimum at $x = \frac{-b}{2a}$ and the minimum value is $\frac{4ac - b^2}{4a}$.

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1. Form a quadratic equation whose roots are 2, 5.

Sol:- Roots are $\alpha = 2, \beta = 5$

$\alpha + \beta = 2 + 5 = 7, \alpha\beta = 2 \times 5 = 10$

Required equation is $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$$\Rightarrow x^2 - 7x + 10 = 0$$

2. Form a quadratic equation whose roots are $\frac{m}{n}, -\frac{n}{m}$ $(m \neq 0, n \neq 0)$.

Sol:- Roots are $\alpha = \frac{m}{n}, \beta = -\frac{n}{m}$

$\alpha + \beta = \frac{m}{n} - \frac{n}{m} = \frac{m^2 - n^2}{mn}$

$\alpha\beta = \left(\frac{m}{n}\right)\left(-\frac{n}{m}\right) = -\frac{mn}{mn} = -1$

Required equation is $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$$\Rightarrow x^2 - \left(\frac{m^2 - n^2}{mn}\right)x - 1 = 0$$

$$\Rightarrow mnx^2 - (m^2 - n^2)x - mn = 0$$

3. Form a quadratic equation whose roots are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

Sol:- Let $\alpha = 2 + \sqrt{3}$ and $\beta = 2 - \sqrt{3}$

Now, $\alpha + \beta = \left(2 + \sqrt{3}\right) + \left(2 - \sqrt{3}\right) = 4$

$\alpha\beta = \left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right) = 4 - 3 = 1$

A quadratic equation having roots α and β is of the form $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$\therefore$ The required equation is $x^2 - 4x + 1 = 0$.

4. Form a quadratic equation whose roots are $2\sqrt{3} - 5$ and $-2\sqrt{3} - 5$.

Sol:- Let $\alpha = 2\sqrt{3} - 5$ and $\beta = -2\sqrt{3} - 5$

Then $\alpha + \beta = 2\sqrt{3} - 5 + \left(-2\sqrt{3} - 5\right) = -10$

and $\alpha\beta = \left(2\sqrt{3} - 5\right)\left(-2\sqrt{3} - 5\right) = -12 + 25 = 13$

Therefore $x^2 - (\alpha+\beta)x + \alpha\beta = 0$ becomes

$x^2 - (-10)x + 13 = 0 \Rightarrow x^2 + 10x + 13 = 0$.

5. Find the roots of the equation $-x^2+x+2=0$.

Sol:- The roots of the quadratic equation $ax^2+b+c=0$ are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Here a=-1, b=1, c=2

$\therefore$ The root of given equation are

$$\frac{-1\pm\sqrt{1^2-4(-1)(2)}}{2(-1)}=\frac{-1\pm\sqrt{1+8}}{-2}$$

$$=\frac{-1\pm\sqrt{9}}{-2}=\frac{-1\pm3}{-2}=\frac{-1+3}{-2}\text{ or }\frac{-1-3}{-2}=-1\text{ or }2$$

6. Find the roots of the equation $4x^2-4x+17=3x^2-10x-17$

Sol:- $4x^2-4x+17=3x^2-10x-17\Rightarrow x^2+6x+34=0$.........(1)

The roots of $ax^2+bx+c=0$ are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Here a = 1, b = 6, c = 34

$\therefore$ The roots of given equation are $\frac{-6\pm\sqrt{36-4(1)(34)}}{2(1)}$

$$=\frac{-6\pm\sqrt{-100}}{2}=\frac{-6\pm10i}{2}$$

$$=-3+5i,\ -3-5i$$

Hence the roots of the given equation are $-3+5i,\ -3-5i$

7. Find the discriminant of the quadratic equation $2x^2-5x+6=0$.

Sol:- Discriminant $\Delta=b^2-4ac=(-5)^2-4(2)(6)$

$$=25-48=-23$$

8. Find the nature of the roots of quadratic equation $x^2-12x+32=0$.

Sol:- For quadratic equation $x^2-12x+32=0$,

$\Delta=b^2-4ac=(-12)^2-4(1)(32)=144-128=16>0$

$\Delta>0\Rightarrow$ the roots of given equation are real and distinct.

Further a = 1, b = -1 and c = 32 are rational and $\Delta=16$ is the square of non-zero rational number 4.

$\therefore$ The roots are distinct rational numbers.

9. If α, β are the roots of the equation $ax^2+bx+c=0$, find the values of the following expressions in terms of a, b, c

(i) $\frac{1}{\alpha}+\frac{1}{\beta}$ **(ii)** $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ **(iii)** $\alpha^2+\beta^2$ **(iv)** $\alpha^3+\beta^3$ **(v)** $\alpha^4\beta^7+\alpha^7\beta^4$

(vi) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2$ **(vii)** $\frac{\alpha^2+\beta^2}{\alpha^{-2}+\beta^{-2}}$

Sol:- From the hypothesis $\alpha+\beta=\frac{-b}{a}$ and $\alpha\beta=\frac{c}{a}$

(i)* $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}=\frac{-b/a}{c/a}=\frac{-b}{c}$

(ii)* $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=\frac{\left(-b/a\right)^2-2\,c/a}{\left(c/a\right)^2}$

$=\frac{\left(b^2/a^2\right)-2\,c/a}{\left(c^2/a^2\right)}=\frac{b^2-2ac}{c^2}$

(iii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2(\alpha\beta)=\left(\frac{-b}{a}\right)^2-2\left(\frac{c}{a}\right)=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$

(iv) $\alpha^3+\beta^3=(\alpha+\beta)\left(\alpha^2+\beta^2-\alpha\beta\right)$

$=(\alpha+\beta)\left((\alpha+\beta)^2-2\alpha\beta-\alpha\beta\right)=(\alpha+\beta)\left((\alpha+\beta)^2-3\alpha\beta\right)$

$=\left(\frac{-b}{a}\right)\left(\left(\frac{-b}{a}\right)^2-3\frac{c}{a}\right)=\frac{-b}{a}\left[\frac{b^2}{a^2}-\frac{3c}{a}\right]=\frac{3abc-b^3}{a^3}$

(v) $\alpha^4\beta^7+\alpha^7\beta^4=(\alpha\beta)^4\left(\alpha^3+\beta^3\right)$

$=\left(\frac{c}{a}\right)^4\left(\frac{3abc-b^3}{a^3}\right)$

$=\frac{bc^4(3ac-b^2)}{a^7}$

(vi) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2=\left(\frac{\alpha^2-\beta^2}{\alpha\beta}\right)^2=\left[\frac{(\alpha+\beta)(\alpha-\beta)}{\alpha\beta}\right]^2=\frac{(\alpha+\beta)^2(\alpha-\beta)^2}{(\alpha\beta)^2}$

$=\frac{\left(-b/a\right)^2}{\left(c/a\right)^2}\left[(\alpha+\beta)^2-4\alpha\beta\right]=\frac{b^2}{c^2}\left[\left(\frac{-b}{a}\right)^2-4\frac{c}{a}\right]=\frac{b^2}{c^2}\left[\frac{b^2-4ac}{a^2}\right]=\frac{b^2\left(b^2-4ac\right)}{a^2c^2}$

(vii) $\dfrac{\alpha^2+\beta^2}{\alpha^{-2}+\beta^{-2}}=\dfrac{\alpha^2+\beta^2}{\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}}=\dfrac{\alpha^2+\beta^2}{\dfrac{\beta^2+\alpha^2}{\alpha^2\beta^2}}=(\alpha\beta)^2=\dfrac{c^2}{a^2}$

10. Prove that the roots of $(x-a)(x-b)=h^2$ are always real.

Sol:- $(x-a)(x-b)=h^2 \Rightarrow x^2-(a+b)x+ab-h^2=0$

$\Delta=\left[-(a+b)\right]^2-4(1)\left(ab-h^2\right)=(a+b)^2-4ab+4h^2=(a-b)^2+(2h)^2>0$

$\Delta>0 \Rightarrow$ roots are always real

6. Find the maximum or minimum of the following expressions as x various over R.

(i) x^2-x+7 and (ii) $2x-7-5x^2$

Sol:- (i) x^2-x+7

Comparing the given expression with ax^2+bx+c

we have a = 1, b = –1, c = 7

So $\dfrac{4ac-b^2}{4a}=\dfrac{4(1)(7)-(-1)^2}{4(1)}=\dfrac{28-1}{4}=\dfrac{27}{4}$

Since $a=1>0$, x^2-x+7 has absolute minimum value $\dfrac{27}{4}$

(ii) $2x-7-5x^2$

Sol:- Comparing the given expression with ax^2+bx+c

we have a = –5, b = 2, c = –7

So $\dfrac{4ac-b^2}{4a}=\dfrac{4(-5)(-7)-(2)^2}{4(-5)}=\dfrac{140-4}{-20}=\dfrac{-136}{20}=\dfrac{-34}{5}$

Since $a=-5<0$, $2x-7-5x^2$ has absolute maximum value $\dfrac{-34}{5}$

Problems for Practice

(i) Find two consecutive positive even intigers, the sum of whose squares is 340.

Ans : 12, 14

(ii) Find the maximum or minimum of the following expressions as x various over R.

(a) $3x^2+2x+11$ Ans:- minimum value $\dfrac{32}{3}$

(b) $12x-x^2-32$ Ans:- maximum value $=4$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

1. For what values of m, the equation $x^2 - 15 - m(2x - 8) = 0$ will have equal roots ?

Sol:- The given equation can be written as $x^2 - 2mx + 8m - 15 = 0$ and it will have equal roots when $\Delta = b^2 - 4ac = 0$

Here $\Delta = 0 \Rightarrow (-2m)^2 - 4(1)(8m - 15) = 0$

$\Rightarrow 4m^2 - 32m + 60 = 0$

$\Rightarrow m^2 - 8m + 15 = 0$

$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3 \text{ or } m = 5$

2. For what values of m, the equation $x^2 + (m+3)x + (m+6) = 0$ will have equal roots ?

Sol:- The given equation, $x^2 + (m+3)x + (m+6) = 0$ will have equal roots when $\Delta = b^2 - 4ac = 0$

Here $\Delta = 0 \Rightarrow (m+3)^2 - 4(1)(m+6) = 0$

$\Rightarrow m^2 + 6m + 9 - 4m - 24 = 0$

$\Rightarrow m^2 + 2m - 15 = 0$

$\Rightarrow (m+5)(m-3) = 0$

$\Rightarrow m = -5 \text{ or } m = 3$

3. If $x^2 - 6x + 5 = 0$ and $x^2 - 12x + p = 0$ have a common root, then find p.

Sol:- Let α be the common root of $x^2 - 6x + 5 = 0$ and $x^2 - 12x + p = 0$

Then $\alpha^2 - 6\alpha + 5 = 0$,

$\Rightarrow (\alpha - 1)(\alpha - 5) = 0 \Rightarrow \alpha = 1 \text{ or } 5$

If $\alpha = 1$ then $\alpha^2 - 12\alpha + p = 0 \Rightarrow 1 - 12 + p = 0 \Rightarrow p = 11$

If $\alpha = 5$ then $\alpha^2 - 12\alpha + p = 0 \Rightarrow 25 - 60 + p = 0 \Rightarrow p = 35$

$\therefore p = 11 \text{ or } 35$

4. If the quadratic equations $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$ $(b \neq c)$ have a common root, then show that $a + 4b + 4c = 0$

Sol:- Let α be the common root of $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$

Then $a\alpha^2 + 2b\alpha + c = 0$............(1) $a\alpha^2 + 2c\alpha + b = 0$...........(2)

$(1) - (2) \Rightarrow (2b - 2c)\alpha + c - b = 0$

$\Rightarrow 2(b-c)\alpha = b - c = 0 \Rightarrow \alpha = \frac{1}{2}$

substitutites $\alpha = \frac{1}{2}$ in (1), we get

$$a\left(\frac{1}{2}\right)^2 + 2b\left(\frac{1}{2}\right) + c = 0 \Rightarrow \frac{a}{4} + b + c = 0 \Rightarrow a + 4b + 4c = 0$$

Problems for Practice

(i) For what values of m, the equation $x^2 - 2(1+3m)x + 7(3+2m) = 0$ will have equal roots?

Ans:- $m = \frac{-10}{9}$ or 2

(ii) If $x^2 - 6x + 5 = 0$ and $x^2 - 3ax + 35 = 0$ have a common root, then find a.

Ans:- a = 4 or 12

LONG ANSWER TYPE QUESTIONS (7 MARKS)

1. Determine the range of the following expressions

(i) $\frac{x^2 + x + 1}{x^2 - x + 1}$

Sol:- Let $y = \frac{x^2 + x + 1}{x^2 - x + 1}$

$$\Rightarrow y(x^2 - x + 1) = x^2 + x + 1$$

$$\Rightarrow x^2y - xy + y = x^2 + x + 1$$

$$\Rightarrow (y-1)x^2 - (y+1)x + y - 1 = 0$$

$$\because x \in R,\ \Delta = b^2 - 4ac \geq 0$$

$$\Rightarrow \left[-(y+1)\right]^2 - 4(y-1)(y-1) \geq 0$$

$$\Rightarrow (y+1)^2 - 4(y-1)^2 \geq 0$$

$$\Rightarrow y^2 + 2y + 1 - 4\left[y^2 - 2y + 1\right] \geq 0$$

$$\Rightarrow -3y^2 + 10y - 3 \geq 0$$

$$\Rightarrow -3y^2 + 9y + y - 3 \geq 0$$

$$\Rightarrow -3y(y-3) + 1(y-3) \geq 0$$

$$\Rightarrow (y-3)(1-3y) \geq 0 \Rightarrow \frac{1}{3} \leq y \leq 3 \qquad \left(\because a = \text{coefficient of } y^2 = -3 < 0\right)$$

$\Rightarrow y \in \left[\frac{1}{3}, 3\right]$

$\therefore$ Range of $\frac{x^2+x+1}{x^2-x+1}$ is $\left[\frac{1}{3}, 3\right]$

2. Prove that $\frac{1}{3x+1}+\frac{1}{x+1}-\frac{1}{(3x+1)(x+1)}$ does not lie between 1 and 4, if x is real.

Sol:- Let $y=\frac{1}{3x+1}+\frac{1}{x+1}-\frac{1}{(3x+1)(x+1)}$

$\Rightarrow y=\frac{x+1+3x+1-1}{(3x+1)(x+1)}=\frac{4x+1}{3x^2+4x+1}$

$\Rightarrow y(3x^2+4x+1)=4x+1$

$\Rightarrow 3yx^2+4xy+y=4x+1$

$\Rightarrow 3yx^2+4(y-1)x+(y-1)=0$

$\because x \in R \Rightarrow \Delta=b^2-4ac \geq 0$

$\Rightarrow [4(y-1)]^2-4(3y)(y-1) \geq 0$

$\Rightarrow 16(y-1)^2-12y(y-1) \geq 0$

$\Rightarrow 4(y-1)[4(y-1)-3y] \geq 0$

$\Rightarrow 4(y-1)[4y-4-3y] \geq 0$

$\Rightarrow 4(y-1)(y-4) \geq 0$

$\Rightarrow (y-1)(y-4) \geq 0 \Rightarrow y \leq 1$ or $y \geq 4$ $\quad$ ($\because$ a = y^2 coefficient = 1 > 0)

$\Rightarrow$ y does not lie between 1 and 4.

3. If x is real, Prove that $\frac{x}{x^2-5x+9}$ lies between $\frac{-1}{11}$ and 1

Sol:- $\frac{x}{x^2-5x+9}=y$

$\Rightarrow\ x = yx^2 - 5yx + 9y$

$\Rightarrow yx^2 + (-5y-1)x + 9y = 0$

$\Rightarrow yx^2 + (-5y-1)x + 9y = 0$

$\because x \in R \Rightarrow \Delta = b^2 - 4ac \geq 0$

$\Rightarrow (-5y-1)^2 - 4(y)(9y) \geq 0$

$\Rightarrow 25y^2 + 1 + 10y - 36y^2 \geq 0$

$\Rightarrow -11y^2 + 10y + 1 \geq 0$

$\Rightarrow 11y^2 - 10y - 1 \leq 0$

$\Rightarrow 11y^2 - 11y + y - 1 \leq 0$

$\Rightarrow (11y+1)(y-1) \leq 0$

$\Rightarrow -\frac{1}{11} \leq y \leq 1$

$\therefore$ y lies between $-\frac{1}{11}$ and 1.

4. If the expression $\frac{x - p}{x^2 - 3x + 2}$ takes all real values for $x \in R$, then find the bounds for p.

Sol:- Let $y = \frac{x-p}{x^2-3x+2}$

$\Rightarrow x^2y - 3xy + 2y = x - p$

$\Rightarrow x^2y - (3y+1)x + (2y+p) = 0$

$\because x \in R \Rightarrow \Delta = b^2 - 4ac \geq 0$

$\Rightarrow [-(3y+1)]^2 - 4y(2y+p) \geq 0$

$\Rightarrow 9y^2 + 6y + 1 - 8y^2 - 4py \geq 0$

$\Rightarrow y^2 + 2(3-2p)y + 1 \geq 0$

a = 1>0, expression is always positive $\Rightarrow$ roots are non real complex numbers

$\Rightarrow b^2 - 4ac < 0$

$\Rightarrow [2(3-2p)]^2 - 4(1)(1) < 0$

$\Rightarrow 4(3-2p)^2 - 4 < 0$

$\Rightarrow (3-2p)^2 - 1 < 0$

$\Rightarrow 9 + 4p^2 - 12p - 1 < 0$

$\Rightarrow 4p^2 - 12p + 8 < 0$

$\Rightarrow p^2 - 3p + 2 < 0$

$\Rightarrow (p-1)(p-2) < 0$

$\Rightarrow$ P lies between 1 and 2

i.e., $1 < p < 2$

5. Solve the equation $4^{x-1} - 3.2^{x-1} + 2 = 0$.

Sol:- Let $2^{x-1} = a$

Then $4^{x-1} - 3.2^{x-1} + 2 = 0$

$\Rightarrow (2^2)^{x-1} - 3.2^{x-1} + 2 = 0$

$\Rightarrow (2^{x-1})^2 - 3.2^{x-1} + 2 = 0$

$\Rightarrow a^2 - 3a + 2 = 0$

$\Rightarrow (a-1)(a-2) = 0$

$\Rightarrow a = 1 \text{ or } 2$

Case (i) If a = 1

Then $2^{x-1} = 1 = 2^0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

Case (ii) If a = 2

Then $2^{x-1} = 2 = 2^1 \Rightarrow x-1=1 \Rightarrow x=2$

$\therefore x = 1 \text{ or } 2$

6. Solve $\sqrt{\frac{x}{x-3}} + \sqrt{\frac{x-3}{x}} = \frac{5}{2}$, when $x \neq 0$ and $x \neq 3$.

Sol:- Let $\sqrt{\frac{x}{x-3}} = a$

Then the given equation becomes $a + \frac{1}{a} = \frac{5}{2}$

$\Rightarrow \frac{a^2+1}{a} = \frac{5}{2} \Rightarrow 2(a^2+1) = 5a$

$\Rightarrow 2a^2 - 5a + 2 = 0$

$\Rightarrow 2a^2 - 4a - a + 2 = 0$

$\Rightarrow 2a(a-2) - 1(a-2) = 0$

$\Rightarrow (a-2)(2a-1) = 0$

$\Rightarrow a = 2 \text{ or } a = \frac{1}{2}$

Case (i) If a = 2

then $\sqrt{\frac{x}{x-3}} = 2 \Rightarrow \frac{x}{x-3} = 4$

$\Rightarrow x = 4x - 12$

$\Rightarrow 3x = 12 \Rightarrow x = 4$

Case (ii) If $a = \frac{1}{2}$ then $\sqrt{\frac{x}{x-3}} = \frac{1}{2} \Rightarrow \frac{x}{x-3} = \frac{1}{4}$

$\Rightarrow 4x = x - 3 \Rightarrow 3x = -3 \Rightarrow x = -1$

$\therefore x = -1 \text{ or } 4$

7. Solve $2\left(x + \frac{1}{x}\right)^2 - 7\left(x + \frac{1}{x}\right) + 5 = 0$ when $x \neq 0$.

Sol:- Let $\left(x + \frac{1}{x}\right) = a$, then the given equation reduced to $2a^2 - 7a + 5 = 0$

$\Rightarrow 2a^2 - 2a - 5a + 5 = 0$

$\Rightarrow 2a(a-1) - 5(a-1) = 0$

$\Rightarrow (a-1)(2a-5) = 0 \Rightarrow a = 1 \text{ or } \frac{5}{2}$

Case (i) :- If a = 1

then $x + \frac{1}{x} = 1 \Rightarrow x^2 - x + 1 = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

Case (ii) :- If $a = \frac{5}{2}$

then $x + \frac{1}{x} = \frac{5}{2} \Rightarrow \frac{x^2+1}{x} = \frac{5}{2} \Rightarrow 2x^2 - 5x + 2 = 0$

$\Rightarrow 2x^2 - 4x - x + 2 = 0 \Rightarrow 2x(x-2) - 1(x-2) = 0$

$\Rightarrow (x-2)(2x-1) = 0$

$\Rightarrow x = \frac{1}{2} \text{or } 2$

$\therefore x = \frac{1}{2}, \frac{1 \pm i\sqrt{3}}{2}, 2$

8. Solve $\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0$, when $x \neq 0$.

Sol:- $\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0$

$$\left[\left(x + \frac{1}{x}\right)^2 - 2\right] - 5\left(x + \frac{1}{x}\right) + 6 = 0$$

Let us take $x + \frac{1}{x} = a$

$\Rightarrow (a^2 - 2) - 5a + 6 = 0$

$\Rightarrow a^2 - 5a + 4 = 0$

$\Rightarrow (a-1)(a-4) = 0$

$\Rightarrow a = 1 \text{ or } 4$

$\Rightarrow x+\dfrac{1}{x}=1$ or $x+\dfrac{1}{x}=4$

$\Rightarrow x^2-x+1=0$ or $x^2-4x+1=0$

$\Rightarrow x=\dfrac{1\pm\sqrt{(1-4)}}{2}$ or $x=\dfrac{4\pm\sqrt{(16-4)}}{2}$

$\Rightarrow x=\dfrac{1\pm\sqrt{3}\,i}{2}$ or $x=2\pm\sqrt{3}$

9. If $c^2 \neq ab$ and the roots of $(c^2-ab)x^2-2(a^2-bc)x+(b^2-ac)=0$ are equal, then show that $a^3+b^3+c^3=3abc$ or $a=0$

Sol:- Discriminant, $\Delta=b^2-4ac=0.$

$\left[-2(a^2-bc)\right]^2-4(c^2-ab)(b^2-ac)=0$

$\Leftrightarrow (a^2-bc)^2=(c^2-ab)(b^2-ac)$

$\Leftrightarrow a^4+b^2c^2-2a^2bc=b^2c^2-ab^3-ac^3+a^2bc$

$\Leftrightarrow a(a^3+b^3+c^3)=3a(abc)$

$\Leftrightarrow a^3+b^3+c^3=3abc$ or $a=0$.

Problems for Practice

(i) Determine the range of $\dfrac{x+2}{2x^2+3x+6}$ Ans: $\left[-\dfrac{1}{13},\dfrac{1}{3}\right]$

(ii) Determine the range of $\dfrac{2x^2-6x+5}{x^2-3x+2}$ Ans: $(-\infty,-2]\cup[2,\infty)$

(iii) Solve $3^{1+x}+3^{1-x}=10$ Ans: $x=-1$ or 1

(iv) Solve $7^{1+x}+7^{1-x}=50$ for real x Ans: $x=-1$ or 1

(v) Solve $x^{\frac{2}{3}}+x^{\frac{1}{3}}-2=0$ Ans: $x=-8$ or 1

(vi) Solve $\sqrt{\dfrac{3x}{x+1}}+\sqrt{\dfrac{x+1}{3x}}=2$, when $x\neq 0$ and $x\neq -1$ Ans: $x=\dfrac{1}{2}$

* * * * *

Unit **4**

Theory of Equations

➪ If $f(x)$ is a polynomial of degree $n > 0$, then the equation $f(x) = 0$ is called Algebraic equation of degree 'n'.

➪ $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots\dots + a_n$ where $a_0, a_1, a_2, \dots\dots\dots a_n \in R \text{ or } C$ and $a_0 \neq 0$ is called a Polynomial in 'x' of degree 'n'.

➪ If a complex number 'α' is such that $f(\alpha) = 0$, then 'α' is called root of the equation $f(x) = 0$.

➪ Relation between the roots and co-efficients of an equation :

(i) If $\alpha_1, \alpha_2, \alpha_3$ are the roots of cubic equation $x^3 + p_1x^2 + p_2x + p_3 = 0$, then

$$S_1 = \alpha_1 + \alpha_2 + \alpha_3 = -p_1$$
$$S_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = p_2$$
$$S_3 = \alpha_1\alpha_2\alpha_3 = -p_3$$

(ii) If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of biquadratic equation $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$, then

$$S_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -p_1$$
$$S_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_4 = p_2$$
$$S_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_1 + \alpha_1\alpha_2\alpha_4 = -p_3$$
$$S_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = p_4$$

➪ For a **cubic equation**, if the roots are

(i) in A.P., then they are taken as a-d, a, a+d

(ii) in G.P., then they are taken as $\frac{a}{r}, a, ar$

(iii) in H.P., then they are taken as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

➪ **Transformation of equations**

(i) Roots with change of sign :

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are roots of $f(x) = 0$, then $-\alpha_1, -\alpha_2, -\alpha_3, \dots, -\alpha_n$ are the roots of $f(-x) = 0$

(ii) Roots multiplied by a given number :

If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are roots of $f(x)=0$, then for any non-zero complex number 'k', the roots of $f(\frac{x}{k})=0$ are $k\alpha_1, k\alpha_2, ..., k\alpha_n$

(iii) If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are roots of the polynomial equation $f(x)=0$, then $\alpha_1 - h, \alpha_2 - h, ..., \alpha_n - h$ are the roots of the equation $f(x+h)=0$

(iv) If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are roots of the polynomial equation $f(x)=0$, then $\alpha_1 + h, \alpha_2 + h, ..., \alpha_n + h$ are roots of the polynomial equation $f(x-h)=0$

(v) Reciprocal roots

If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are the roots of polynomial equation $f(x)=0$, then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ..., \frac{1}{\alpha_n}$ are the roots of the equation $x^n . f\left(\frac{1}{x}\right)=0$

(vi) If 'α' is a root of $f(x)=0$, then α^2 is a root of $f\left(\sqrt{x}\right)=0$

Note :- If an equation is unaltered by changing 'x' into $\frac{1}{x}$, then it is a **reciprocal equation**

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1. If 1, 1, α are the roots of $x^3 - 6x^2 + 9x - 4 = 0$, then find 'α'

Sol:- $x^3 - 6x^2 + 9x - 4 = 0$

$\left(a_0 = 1, a_1 = -6, a_2 = 9, a_3 = -4\right)$

$S_1 = \frac{-a_1}{a_0} \Rightarrow S_1 = 1 + 1 + \alpha = \frac{-(-6)}{1}$

$\Rightarrow 2 + \alpha = 6$

$\Rightarrow \alpha = 6 - 2 = 4$

$\therefore \alpha = 4$

2. If -1, 2 and α are the roots of $2x^3 + x^2 - 7x - 6 = 0$, then find 'α'.

Sol:- $2x^3 + x^2 - 7x - 6 = 0$

$\left(a_0 = 2, a_1 = 1, a_2 = -7, a_3 = -6\right)$

$S_1 = \frac{-a_1}{a_0} \Rightarrow S_1 = -1 + 2 + \alpha = \frac{-1}{2}$

$\Rightarrow \alpha + 1 = \frac{-1}{2}$

$\Rightarrow \alpha = \frac{-1}{2} - 1 = \frac{-3}{2}$

$\therefore \alpha = \frac{-3}{2}$

3. If 1, -2 and 3 are the roots of $x^3 - 2x^2 + ax + 6 = 0$, then find 'a'.

Sol:- $x^3 - 2x^2 + ax + 6 = 0$

$(a_0 = 1, a_1 = -2, a_2 = a, a_3 = 6)$

$S_2 = \frac{a_2}{a_0} \Rightarrow \alpha\beta + \alpha\gamma + \beta\gamma = \frac{a}{1}$

$(1)(-2) + (1)(3) + (-2)(3) = a$

$-2 + 3 - 6 = a$

$\therefore a = -5$

4. If α, β and 1 are the roots of $x^3 - 2x^2 - 5x + 6 = 0$, then find α and β.

Sol:- $x^3 - 2x^2 - 5x + 6 = 0$

$(a_0 = 1, a_1 = -2, a_2 = -5, a_3 = 6)$

$S_1 = \frac{-a_1}{a_0} \Rightarrow \alpha + \beta + 1 = \frac{-(-2)}{1}$

$\Rightarrow \alpha + \beta = 1$ ....(1)

$S_3 = \frac{-a_3}{a_0} \Rightarrow (\alpha)(\beta)(1) = \frac{-6}{1}$

$\Rightarrow \alpha\beta = -6$ ....(2)

From (1), $\beta = 1 - \alpha$, Substituting in equation (2)

$\alpha(1-\alpha) = -6$

$\alpha - \alpha^2 = -6$

$-\alpha^2 + \alpha + 6 = 0$

$\alpha^2 - \alpha - 6 = 0$

$\alpha^2 - 3\alpha + 2\alpha - 6 = 0$

$\alpha(\alpha - 3) + 2(\alpha - 3) = 0$

$(\alpha - 3)(\alpha + 2) = 0$

$\alpha = -2 \; or \; 3$

5. If the Product of roots of $4x^3 + 16x^2 - 9x - a = 0$ is 9, then find 'a'.

Sol:- $4x^3 + 16x^2 - 9x - a = 0$

$(a_0 = 4, a_1 = 16, a_2 = -9, a_3 = -a)$

Given product of roots $S_3 = \frac{-a_3}{a_0} = 9$

$\Rightarrow \frac{-(-a)}{4} = 9$

$\Rightarrow a = 36$

6. Find the polynomial equation whose roots are negatives of the roots of the equation $x^4 - 6x^3 + 7x^2 - 2x + 1 = 0$

Sol:- If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are roots of $f(x) = 0$, then $-\alpha_1, -\alpha_2, -\alpha_3, ... -\alpha_n$ are the roots of $f(-x) = 0$

$x^4 - 6x^3 + 7x^2 - 2x + 1 = 0$

Put $x = -x$

$(-x)^4 - 6(-x)^3 + 7(-x)^2 - 2(-x) + 1 = 0$

$x^4 + 6x^3 + 7x^2 + 2x + 1 = 0$

7. Find the equation whose roots are the reciprocals of the roots of $\mathbf{x^4 + 3x^3 - 6x^2 + 2x - 4 = 0}$.

Sol:- If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are roots of $f(x) = 0$,

then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, ..., \frac{1}{\alpha_n}$ are the roots of equation $f\left(\frac{1}{x}\right) = 0$

$x^4 + 3x^3 - 6x^2 + 2x - 4 = 0$

Put $x = \frac{1}{x}$

$\left(\frac{1}{x}\right)^4 + 3\left(\frac{1}{x}\right)^3 - 6\left(\frac{1}{x}\right)^2 + \frac{2}{x} - 4 = 0$

$\Rightarrow \frac{1}{x^4} + \frac{3}{x^3} - \frac{6}{x^2} + \frac{2}{x} - 4 = 0$

$\Rightarrow 1 + 3x - 6x^2 + 2x^3 - 4x^4 = 0$

$\Rightarrow -4x^4 + 2x^3 - 6x^2 + 3x + 1 = 0$

$\Rightarrow 4x^4 - 2x^3 + 6x^2 - 3x - 1 = 0$

8. Find the algebraic equation whose roots are 2 times the roots of $\mathbf{x^5 - 2x^4 + 3x^3 - 2x^2 + 4x + 3 = 0}$.

Sol:- If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are the roots of $f(x) = 0$,

then $k\alpha_1, k\alpha_2, k\alpha_3, ..., k\alpha_n$ are the roots of $f\left(\frac{x}{k}\right) = 0$

$x^5 - 2x^4 + 3x^3 - 2x^2 + 4x + 3 = 0$

$\therefore Put\ x = \frac{x}{2}$

$\Rightarrow \left(\frac{x}{2}\right)^5 - 2\left(\frac{x}{2}\right)^4 + 3\left(\frac{x}{2}\right)^3 - 2\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) + 3 = 0$

$\Rightarrow \frac{x^5}{32} - 2\left(\frac{x^4}{16}\right) + 3\left(\frac{x^3}{8}\right) - 2\left(\frac{x^2}{4}\right) + \frac{4x}{2} + 3 = 0$

$$\Rightarrow \frac{x^5 - 4x^4 + 12x^3 - 16x^2 + 64x + 96}{32} = 0$$

$$\Rightarrow x^5 - 4x^4 + 12x^3 - 16x^2 + 64x + 96 = 0$$

Problems for Practice:

(i) Form polynomial equation of lowest degree with roots

(a) 1, -1, 3 Ans: $x^3 - 3x^2 - x + 3 = 0$

(b) $2 \pm \sqrt{3}, 1 \pm 2i$ Ans: $x^4 - 6x^3 + 14x^2 - 22x + 5 = 0$

(c) $0, 1, \frac{-3}{2}, \frac{-5}{2}$ Ans: $4x^4 + 12x^3 - x^2 - 15x = 0$

(ii) If 2 is a root of $x^3 - 6x^2 + 3x + 10 = 0$ then find the other roots.

Ans: –1, 5 (Hint : Let roots be α, β, 2)

(iii) If 1, 2, 3, 4 are the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$, then find the values of a, b, c, d.

Ans: a = –10, b = 35, c = –50, d = 24

(iv) Find the polynomial equation whose roots are thenegatives of the roots of the equation $x^4 + 5x^3 + 11x + 3 = 0$.

Ans: $x^4 - 5x^3 + 11x + 3 = 0$

(v) Find the equation whose roots are the reciprocals of the roots of $x^5 + 11x^4 + x^3 + 4x^2 - 13x + 6 = 0$.

Ans: $6x^5 - 13x^4 + 4x^3 + x^2 + 11x + 1 = 0$

(vi) Find the polynomial equation whose roots are the reciprocals of the roots of $x^4 - 3x^3 + 7x^2 + 5x - 2 = 0$.

Ans: $2x^4 - 5x^3 - 7x^2 + 3x - 1 = 0$

(vii) Find the algebraic equation whose roots are 3 times the roots of $x^3 + 2x^2 - 4x + 1 = 0$.

Ans: $x^3 + 6x^2 - 36x + 27 = 0$ (Hint : put $x = \frac{x}{3}$)

(viii) Find the equation of degree 4 whose roots are 3 times the roots of $6x^4 - 7x^3 + 8x^2 - 7x + 2 = 0$.

Ans: $6x^4 - 21x^3 + 72x^2 - 189x + 162 = 0$ (Refer Text Book Page No. 133. Ex.2)

(ix) Find the equation whose roots are 'm' times the roots of equation $x^3 + \frac{1}{4}x^2 - \frac{1}{16}x + \frac{1}{72} = 0$ and deduce the case if m = 12.

Ans: $x^3 + 3x^2 - 9x + 24 = 0$ (Refer Text Book Page No. 134. Ex.3)

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

1. Solve $x^3 - 3x^2 - 16x + 48 = 0$, given that the sum of two roots is zero.

Sol:- Let $\alpha, -\alpha, \beta$ be the roots of $x^3 - 3x^2 - 16x + 48 = 0$

$(a_0 = 1, a_1 = -3, a_2 = -16, a_3 = 48)$

$$S_1 = \frac{-a_1}{a_0} \Rightarrow \alpha + (-\alpha) + \beta = \frac{-(-3)}{1} \quad \Rightarrow \beta = 3$$

$$S_2 = \frac{a_2}{a_0} \Rightarrow \alpha(-\alpha) + (-\alpha)\beta + \beta\alpha = -16$$

$$\Rightarrow -\alpha^2 - \alpha\beta + \alpha\beta = -16$$

$$\Rightarrow \alpha^2 = 16$$

$$\Rightarrow \alpha = 4$$

Roots are $\alpha, -\alpha, \beta$ i.e., 4, -4, 3

2. Solve the equation $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$, given that $1 + i$ is one of its roots.

Sol:- Given 1+i is one root

$\Rightarrow 1 - i$ is another root.

$\therefore$ The equation having roots $1 \pm i$ is

$[x - (1+i)][x - (1-i)] = 0$

$\Rightarrow [(x-1) - i][(x-1) + i] = 0$

$\Rightarrow (x-1)^2 - i^2 = 0$

$\Rightarrow x^2 - 2x + 1 + 1 = 0$

$\Rightarrow x^2 - 2x + 2 = 0$

$\therefore x^2 - 2x + 2$ is a factor of $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$

	1	2	–5	6	2
2	0	2	8	2	0
–2	0	0	–2	–8	–2
	1	4	1	0	0

$\Rightarrow x^2 + 4x + 1 = 0$

$$\Rightarrow x = \frac{-4 \pm \sqrt{16 - 4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2}$$

$\Rightarrow x = \dfrac{2(-2 \pm \sqrt{3})}{2}$

$\Rightarrow x = (-2 \pm \sqrt{3})$

$\therefore$ The roots of given equation are $\left\{1 \pm i,\ -2 \pm \sqrt{3}\right\}$

3. Solve the equation $x^4 - 6x^3 + 11x^2 - 10x + 2 = 0$, given that $2+\sqrt{3}$ is a root of the equation.

Sol:- Given that $2+\sqrt{3}$ is one root.

$\Rightarrow 2-\sqrt{3}$ is another root.

$\therefore$ The equation having roots $2 \pm \sqrt{3}$ is

$\left[x-(2+\sqrt{3})\right]\left[x-(2-\sqrt{3})\right] = 0$

$\left[(x-2)-\sqrt{3})\right]\left[(x-2)+\sqrt{3})\right] = 0$

$\Rightarrow (x-2)^2 - \left(\sqrt{3}\right)^2 = 0$

$\Rightarrow x^2 - 4x + 4 - 3 = 0$

$\Rightarrow x^2 - 4x + 1 = 0$

$\therefore x^2 - 4x + 1 = 0$ is a factor of $x^4 - 6x^3 + 11x^2 - 10x + 2 = 0$

	1	–6	11	–10	2
4	0	4	–8	8	0
–1	0	0	–1	2	–2
	1	–2	2	0	0

$x^2 - 2x + 2 = 0$

$\Rightarrow x = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2}$

$\Rightarrow x = 1 \pm i$

$\therefore$ The roots of given equation are $\left\{2+\sqrt{3},\ 2-\sqrt{3},\ 1+i,\ 1-i\right\}$

4. Given that $-2+\sqrt{-7}$ is a root of the equation $x^4 + 2x^2 - 16x + 77 = 0$, Solve it completely.

Sol:- Given $-2 + i\sqrt{7}$ is one root

$\Rightarrow -2 - i\sqrt{7}$ is another root

$\therefore$ The equation having roots $-2 \pm i\sqrt{7}$ is

$\left[x-(-2+i\sqrt{7})\right]\left[x-(-2-i\sqrt{7})\right]=0$

$\Rightarrow \left[(x+2)-i\sqrt{7})\right]\left[x+2+i\sqrt{7})\right]=0$

$\Rightarrow (x+2)^2-(i\sqrt{7})^2=0$

$\Rightarrow x^2+4x+4+7=0$

$\Rightarrow x^2+4x+11=0$

$\therefore x^2+4x+11=0$ is a factor of $x^4+2x^2-16x+77=0$

	1	0	2	–16	77
– 4	0	–4	16	–28	0
–11	0	0	–11	44	–77
	1	–4	7	0	0

$\therefore x^2-4x+7=0$

$x=\dfrac{4\pm\sqrt{16-28}}{2}=\dfrac{4\pm i\sqrt{12}}{2}=\dfrac{2(2\pm i\sqrt{3})}{2}$

$\therefore x=(2\pm i\sqrt{3})$

$\therefore$ The roots of given equation are $\left\{2+i\sqrt{3},\ 2-i\sqrt{3},\ -2+i\sqrt{7},\ -2-i\sqrt{7}\right\}$

Problems for Practice

(i) Given that sum of two roots of $x^4-2x^3+4x^2+6x-21=0$ is zero. Find the roots of the equation ?

Ans: $\sqrt{3},\ -\sqrt{3},\ 1+i\sqrt{6}, 1-i\sqrt{6}$

(ii) Given that two roots of $4x^3+20x^2-23x+6=0$ are equal, find all the roots of given equation?

Ans:- $\left\{\frac{1}{2}, \frac{1}{2}, -6\right\}$ (Hint : Let the roots be α, α, β)

(iii) Solve $9x^3-15x^2+7x-1=0$, given that two of its roots are equal.

Ans:- $\left\{\frac{1}{3}, \frac{1}{3}, 1\right\}$

LONG ANSWER TYPE QUESTIONS (7 MARKS)

1. Solve $x^3-9x^2+14x+24=0$, given that two of its roots are in the ratio 3:2.

Sol:- $x^3-9x^2+14x+24=0$

$(a_0=1,\ a_1=-9,\ a_2=14, a_3=24)$

Let the roots be $3\alpha, 2\alpha, \beta$

$$S_1 = \frac{-a_1}{a_0} \Rightarrow 3\alpha + 2\alpha + \beta = -(-9) \Rightarrow 5\alpha + \beta = 9 \text{..............................} (I)$$

$$S_2 = \frac{a_2}{a_0} \Rightarrow (3\alpha)(2\alpha) + (2\alpha)(\beta) + (\beta)(3\alpha) = 14$$

$$\Rightarrow 6\alpha^2 + 2\alpha\beta + 3\alpha\beta = 14$$

$$\Rightarrow 6\alpha^2 + 5\alpha\beta = 14 \text{..............................} (II)$$

Substitute $\beta = 9 - 5\alpha$ in equation II

$$6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$$

$$\Rightarrow 6\alpha^2 + 45\alpha - 25\alpha^2 - 14 = 0$$

$$\Rightarrow 19\alpha^2 - 45\alpha + 14 = 0$$

$$\Rightarrow 19\alpha^2 - 38\alpha - 7\alpha + 14 = 0$$

$$\Rightarrow 19\alpha(\alpha - 2) - 7(\alpha - 2) = 0$$

$$\Rightarrow (19\alpha - 7)(\alpha - 2) = 0$$

$$\Rightarrow \alpha = \frac{7}{19}, \quad \alpha = 2$$

Put $\alpha = 2$ in eqn. (I)

$$5(2) + \beta = 9 \Rightarrow \beta = -1$$

Roots are $3\alpha, 2\alpha, \beta$

$$\Rightarrow 3(2), 2(2), -1$$

$$\Rightarrow \{6, 4, -1\}$$

2. Solve the equation $8x^3 - 36x^2 - 18x + 81 = 0$ if the roots are in A.P.

Sol:- $8x^3 - 36x^2 - 18x + 81 = 0$

$(a_0 = 8, a_1 = -36, a_2 = -18, a_3 = 81)$

Let the roots in A.P. be a-d, a, a+d

$$S_1 = \frac{-a_1}{a_0} \Rightarrow (a-d) + (a) + (a+d) = \frac{-(-36)}{8}$$

$$\Rightarrow 3a = \frac{9}{2} \qquad \Rightarrow a = \frac{3}{2}$$

$$S_3 = \frac{-a_3}{a_1} \Rightarrow (a-d)(a)(a+d) = \frac{-81}{8}$$

$\Rightarrow a(a^2 - d^2) = \frac{-81}{8}$

Substituting $a = \frac{3}{2}$

$\frac{3}{2}\left[\frac{9}{4} - d^2\right] = \frac{-81}{8}$

$\Rightarrow \frac{3}{2}\left[\frac{9-4d^2}{4}\right] = \frac{-81}{8}$

$\Rightarrow \frac{3(9-4d^2)}{8} = \frac{-81}{8}$

$\Rightarrow (9-4d^2) = -27$

$\Rightarrow -4d^2 = -27-9$

$\Rightarrow 4d^2 = 36$

$\Rightarrow d = \pm 3$

Substituting $a = \frac{3}{2},\ d = 3$

the roots are a-d, a, a+d

$\Rightarrow \frac{3}{2} - 3,\ \frac{3}{2},\ \frac{3}{2} + 3$

$\Rightarrow \left\{\frac{-3}{2},\ \frac{3}{2},\ \frac{9}{2}\right\}$

3. Solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$ if its roots are in G.P.

Sol:- Since the roots are in G.P., they must be of the form *a/r, a, ar.*

$\therefore \left(\frac{a}{r}\right)a(ar) = \frac{24}{3} \Rightarrow a^3 = 8 \Rightarrow a = 2$

$\frac{a}{r} + a + ar = \frac{26}{3} \Rightarrow 2\left(\frac{1}{r} + 1 + r\right) = \frac{26}{3}$

$\Rightarrow \frac{1}{r} + 1 + r = \frac{13}{3} \Rightarrow \frac{1}{r} + r = \frac{10}{3}$

$\Rightarrow r = 3$

$\therefore$ The roots are 2/3, 2, 6.

4. Solve the equation $54x^3 - 39x^2 - 26x + 16 = 0$, given that its roots are in G.P.

Sol:- $54x^3 - 39x^2 - 26x + 16 = 0$

$(a_0 = 54,\ a_1 = -39,\ a_2 = -26,\ a_3 = 16)$

Let the roots in G.P. be $\frac{a}{r},\ a,\ ar$

$S_1 = \frac{-a_1}{a_0} \Rightarrow \frac{a}{r} + a + ar = \frac{-(-39)}{54}$

$\Rightarrow a\left[\frac{1}{r} + 1 + r\right] = \frac{13}{18}$.....................$(I)$

$S_3 = \frac{-a_3}{a_0} \Rightarrow \left(\frac{a}{r}\right)(a)(ar) = \frac{-16}{54}$

$\Rightarrow a^3 = \frac{-8}{27} \Rightarrow a^3 = \left(\frac{-2}{3}\right)^3 \Rightarrow a = \frac{-2}{3}$

Substituting $a = \frac{-2}{3}$ in eqn. I

$\frac{-2}{3}\left[\frac{1 + r + r^2}{r}\right] = \frac{13}{18}$, Cross multiplying

$\Rightarrow -12 - 12r - 12r^2 - 13r = 0$

$\Rightarrow 12r^2 + 25r + 12 = 0$

$\Rightarrow 12r^2 + 16r + 9r + 12 = 0$

$\Rightarrow 4r(3r + 4) + 3(3r + 4) = 0$

$\Rightarrow (4r + 3)(3r + 4) = 0$

$\Rightarrow r = \frac{-3}{4}\ or\ r = \frac{-4}{3}$

Substituting $a = \frac{-2}{3},\ r = \frac{-3}{4}$, the roots are $\frac{a}{r},\ a,\ ar$

$\Rightarrow \frac{-2}{3}\Big/\frac{-3}{4},\ \frac{-2}{3},\ \left(\frac{-2}{3}\right)\left(\frac{-3}{4}\right)$

$\Rightarrow \left\{\frac{8}{9}, \frac{-2}{3}, \frac{1}{2}\right\}$

5. Transform the equation $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$ in which the co-efficient of the second highest power of x is zero and also find its transformed equation.

Sol:- Let $f(x) = x^4 + 4x^3 + 2x^2 - 4x - 2$

we have to find 'h' so that the co-efficient of x^3 in $f(x+h)$ is zero.

we have $f(x+h) = (x+h)^4 + 4(x+h)^3 + 2(x+h)^2 - 4(x+h) - 2$

The co-efficient of x^3 in $f(x+h)$ is ${}^4C_1(h) + 4 = 4h + 4$

we have to select 'h' such that

$4h + 4 = 0 \Rightarrow h = -1$

$\therefore$ Required equation is $f(x-1) = 0$

i.e., $(x-1)^4 + 4(x-1)^3 + 2(x-1)^2 - 4(x-1) - 2 = 0$

–1	1	4	2	–4	–2
	0	–1	–3	1	3
	1	3	–1	–3	$1 = A_4$
	0	–1	–2	3	
	1	2	–3	$0 = A_3$	
	0	–1	–1		
	1	1	$-4 = A_2$		
	0	–1			
	$1 = A_0$	$0 = A_1$			

Required equation : $A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4 = 0$

$\therefore x^4 - 4x^2 + 1 = 0$ is the transformed equation.

6. Find the polynomial equation whose roots are the translates of those of the equation $\mathbf{x^4 - 5x^3 + 7x^2 - 17x + 11 = 0}$ by –2.

Sol:- Let $f(x) = x^5 - 5x^3 + 7x^2 - 17x + 11$.

The required equation is f(x+2) = 0.

2	1	–5	7	–17	11
	0	2	–6	2	–30
2	1	–3	1	–15	$-19 = A_4$
	0	2	–2	– 2	
2	1	–1	–1	$-17 = A_3$	
	0	2	2		
2	1	1	$1 = A_2$		
	0	2			
2	1	$3 = A_1$			
	0				
	$1 = A_0$				

By Horner's process $f(x+2) = A_0\,x^4 + A_1\,x^3 + A_2\,x^2 + A_3\,x + A_4$

$$\Rightarrow f(x+2) = x^4 + 3x^3 + x^2 - 17x - 19$$

The required eqaution is $x^4 + 3x^3 + x^2 - 17x - 19 = 0$

7. Find the polynomial equation whose roots are the translates of those of the equation $\mathbf{x^5 - 4x^4 + 3x^2 - 4x + 6 = 0}$ by –3.

Sol:- Let $f(x) = x^5 - 4x^4 + 3x^2 - 4x + 6 = 0$

The required equation is $f(x+3) = 0$.

3	1	–4	0	3	–4	6
	0	3	–3	–9	–18	–66
3	1	–1	–3	–6	–22	$-60 = A_5$
	0	3	6	9	9	
3	1	2	3	3	$-13 = A_4$	
	0	3	15	54		
3	1	5	18	$57 = A_3$		
	0	3	24			
3	1	8	$42 = A_2$			
	0	3				
3	1	$11 = A_1$				
	0					
	$1 = A_0$					

By Horner's process $f(x+3) = A_0 x^5 + A_1 x^4 + A_2 x^3 + A_3 x^2 + A_4 x + A_5$

$$\Rightarrow f(x+3) = x^5 + 11x^4 + 42x^3 + 57x^2 - 13x - 60$$

$\therefore$ The required eqaution is $x^5 + 11x^4 + 42x^3 + 57x^2 - 13x - 60 = 0$.

8. Solve the equation $\mathbf{x^4 - 10x^3 + 26x^2 - 10x + 1 = 0}$.

Ans:- $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

$$x^2\left(x^2 - 10x + 26 - \frac{10}{x} + \frac{1}{x^2}\right) = 0$$

$$\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots * $$

Put $x + \frac{1}{x} = k$ .... (1)

Squaring on both sides $\left(x + \frac{1}{x}\right)^2 = k^2$

$x^2+\frac{1}{x^2}+2=k^2$

$x^2+\frac{1}{x^2}=k^2-2$ ....(2)

Substituting (1) & (2) in * $\left(k^2-2\right)-10k+26=0$

$k^2-10k+24=0$

$k^2-6k-4k+24=0$

$k\left(k-6\right)-4\left(k-6\right)=0$

$\left(k-4\right)\left(k-6\right)=0$

$k-4=0$	$k-6=0$
$x+\frac{1}{x}-4=0$	$x+\frac{1}{x}-6=0$
$x^2+1-4x=0$	$x^2+1-6x=0$
$x^2-4x+1=0$	$x^2-6x+1=0$
$x=\frac{4\pm\sqrt{16-4}}{2}$	$x=\frac{6\pm\sqrt{36-4}}{2}$
$x=\frac{2\left(2\pm\sqrt{3}\right)}{2}$	$x=\frac{2\left(3\pm2\sqrt{2}\right)}{2}$
$x=2\pm\sqrt{3}$	$x=3\pm2\sqrt{2}$

Solutions $\left\{2+\sqrt{3},\ 2-\sqrt{3},\ 3+2\sqrt{2},\ 3-2\sqrt{2}\right\}$

9. Solve the equation $2x^5+x^4-12x^3-12x^2+x+2=0$.

Sol:- The given equation is an odd degree reciprocal equation of class one

$\therefore$ -1 is a root of this equation

$\therefore$ $x+1$ is a factor of $2x^5+x^4-12x^3-12x^2+x+2=0$

-1	2	1	-12	-12	1	2
	0	-2	1	11	1	-2
	2	-1	-11	-1	2	0

$\therefore 2x^4-x^3-11x^2-x+2=0$

$x^2\left(2x^2-x-11-\frac{1}{x}+\frac{2}{x^2}\right)=0$

$\Rightarrow 2\left(x^2+\frac{1}{x^2}\right)-\left(x+\frac{1}{x}\right)-11=0$ .... *

Let $x+\frac{1}{x}=k$..................(I)

Squaring on both sides

$\left(x+\frac{1}{x}\right)^2=k^2$

$x^2+\frac{1}{x^2}+2=k^2$

$x^2+\frac{1}{x^2}=k^2-2$..........................(II)

Substituting I & II in *

$2\left(k^2-2\right)-k-11=0$

$2k^2-4-k-11=0$

$2k^2-k-15=0$

$2k^2-6k+5k-15=0$

$2k(k-3)+5(k-3)=0$

$(k-3)(2k+5)=0$

$k-3=0$

$x+\frac{1}{x}-3=0$

$x^2-3x+1=0$

$x=\frac{3\pm\sqrt{9-4}}{2}$

$x=\frac{3\pm\sqrt{5}}{2}$

$2k+5=0$

$2\left(x+\frac{1}{x}\right)+5=0$

$2x+\frac{2}{x}+5=0$

$2x^2+5x+2=0$

$2x^2+4x+x+2=0$

$2x(x+2)+1(x+2)=0$

$(x+2)(2x+1)=0$

$x=\frac{-1}{2},-2$

Solutions $\left\{-1,-2,\frac{-1}{2},\frac{3+\sqrt{5}}{2},\frac{3-\sqrt{5}}{2}\right\}$

10. Solve the equation $\mathbf{x^5-5x^4+9x^3-9x^2+5x-1=0}$.

Sol:- The given equation is an odd degree reciprocal equation of class two.

$\Rightarrow$ '1' is a root of given equation

$\therefore x-1$ is a factor of given equation

x=1	1	–5	9	–9	5	–1
	0	1	–4	5	–4	1
	1	–4	5	–4	1	0

$\therefore x^4-4x^3+5x^2-4x+1=0$

$x^2\left(x^2-4x+5-\frac{4}{x}+\frac{1}{x^2}\right)=0$

$\Rightarrow\left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)=0$

Let $x+\frac{1}{x}=k$, Squaring on both sides

$\left(x+\frac{1}{x}\right)^2=k^2$

$x^2+\frac{1}{x^2}+2=k^2$

$x^2+\frac{1}{x^2}=k^2-2$

$\therefore\left(k^2-2\right)-4k+5=0$

$k^2-4k+3=0$

$k^2-3k-k+3=0$

$k(k-3)-1(k-3)=0$

$(k-3)(k-1)=0$

$k-3=0$

$x+\frac{1}{x}-3=0$

$x^2-3x+1=0$

$x=\frac{3\pm\sqrt{9-4}}{2}$

$x=\frac{3\pm\sqrt{5}}{2}$

$k-1=0$

$\left(x+\frac{1}{x}\right)-1=0$

$x^2-x+1=0$

$x=\frac{1\pm i\sqrt{3}}{2}$

Solutions are $\left\{\frac{1+i\sqrt{3}}{2},\frac{1-i\sqrt{3}}{2},\frac{3+\sqrt{5}}{2},\frac{3-\sqrt{5}}{2}\right\}$

11. Solve $6x^6-25x^5+31x^4-31x^2+25x-6=0$

Sol:- The given equation is an even degree reciprocal equation of class two.

$\therefore$ +1 and -1 are the roots of this equation

$\therefore\ x+1$ and $x-1$ are factors of given equation.

x = –1	6	–25	31	0	–31	25	–6
	0	– 6	31	–62	62	–31	6
	6	–31	62	–62	31	–6	0
	0	6	–25	37	–25	6	
	6	–25	37	–25	6	0	

$$\therefore 6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$$

On dividing both sides by x^2, we get

$$6x^2 - 25x + 37 - \frac{25}{x} + \frac{6}{x^2} = 0$$

$$6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0$$

$$Let\ x + \frac{1}{x} = k$$

Squaring on both sides

$$\left(x + \frac{1}{x}\right)^2 = k^2$$

$$x^2 + \frac{1}{x^2} + 2 = k^2$$

$$x^2 + \frac{1}{x^2} = k^2 - 2$$

$$\therefore 6(k^2 - 2) - 25k + 37 = 0$$

$$6k^2 - 25k + 25 = 0$$

$$6k^2 - 15k - 10k + 25 = 0$$

$$3k(2k-5) - 5(2k-5) = 0$$

$$(2k-5)(3k-5) = 0$$

$$2k - 5 = 0 \qquad 3k - 5 = 0$$

$$2\left(x + \frac{1}{x}\right) - 5 = 0 \qquad 3\left(x + \frac{1}{x}\right) - 5 = 0$$

$$2x + \frac{2}{x} - 5 = 0 \qquad 3x + \frac{3}{x} - 5 = 0$$

$$2x^2 - 5x + 2 = 0 \qquad 3x^2 - 5x + 3 = 0$$

$$2x^2 - 4x - x + 2 = 0 \qquad x = \frac{5 \pm \sqrt{25-36}}{6}$$

$2x(x-2)-1(x-2)=0$ $\quad\quad x=\dfrac{5\pm\sqrt{-11}}{6}$

$(x-2)(2x-1)=0$ $\quad\quad x=\dfrac{5\pm i\sqrt{11}}{6}$

$x=\dfrac{1}{2},\ 2$

$\therefore$ Solutions $\left\{\pm 1,\ \dfrac{1}{2},\ 2,\ \dfrac{5\pm i\sqrt{11}}{2}\right\}$

12. Solve the equation $x^4-10x^3+26x^2-10x+1=0$

Sol:- Degree = 4

It is an even degree polynomial equation of class one.

$\therefore$ Divide the equation by both sides $\dfrac{x^4-10x^3+26x^2-10x+1}{x^2}=\dfrac{0}{x^2}$

$\Rightarrow \dfrac{x^4}{x^2}-\dfrac{10x^3}{x^2}+\dfrac{26x^2}{x^2}-\dfrac{10x}{x^2}+\dfrac{1}{x^2}=0$

$\Rightarrow x^2-10x+26-\dfrac{10}{x}+\dfrac{1}{x^2}=0$

$\Rightarrow \left(x^2+\dfrac{1}{x^2}\right)-10\left(x+\dfrac{1}{x}\right)+26=0$................*

Put $x+\dfrac{1}{x}=k$..................(1)

Squaring on both sides

$\left(x+\dfrac{1}{x}\right)^2=k^2$

$\Rightarrow x^2+\dfrac{1}{x^2}+2=k^2$

$\Rightarrow x^2+\dfrac{1}{x^2}=k^2-2$

Substituting (1) and (2) in equation *

$(k^2-2)-10k+26=0$

$k^2-10k+24=0$

$k^2-6k-4k+24=0$

$k(k-6)-4(k-6)=0$

$(k-4)(k-6)=0$

$k-4=0$

$\Rightarrow x+\frac{1}{x}-4=0$

$\Rightarrow \frac{x^2+1-4x}{x}=0$

$\Rightarrow x^2-4x+1=0$

(a = 1, b = -4, c = 1)

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$

$x=\frac{4\pm\sqrt{12}}{2}$

$x=\frac{4\pm2\sqrt{3}}{2}$

$=\frac{2(2\pm\sqrt{3})}{2}$

$=2\pm\sqrt{3}$

$=2+\sqrt{3},\ 2-\sqrt{3}$

$k-6=0$

$\Rightarrow x+\frac{1}{x}-6=0$

$\Rightarrow \frac{x^2+1-6x}{x}=0$

$\Rightarrow x^2-6x+1=0$

(a = 1, b = -6, c = 1)

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(1)}}{2(1)}$

$x=\frac{6\pm\sqrt{36-4}}{2}$

$x=\frac{6\pm\sqrt{32}}{2}$

$=\frac{6\pm4\sqrt{2}}{2}$

$=\frac{2\left(3\pm2\sqrt{2}\right)}{2}$

$=3\pm2\sqrt{2}$

$=3+2\sqrt{2},\ 3-2\sqrt{2}$

Solutions are $\left\{2+\sqrt{3},\ 2-\sqrt{3},\ 3+2\sqrt{2}, 3-2\sqrt{2}\right\}$

Problems for Practice

(i) Given that one root of $2x^3+3x^2-8x+3=0$ is double the other root, find the roots of the equation.

Ans:- $\left\{\frac{1}{2},\ 1,\ -3\right\}$ (Hint : Let the roots be α, 2α, β)

(ii) Solve $x^3-7x^2+36=0$, given one root being twice the other

Ans:- $\{3,\ 6,\ -2\}$

(iii) Solve the equation $x^3-3x^2-6x+8=0$ if the roots are in A.P.

Ans:- { 4, 1, -2 }

(iv) Solve the equation $4x^3-24x^2+23x+18=0$, given that its roots are in A.P.

Ans:- $\left\{\frac{-1}{2},\ 2,\ \frac{9}{2}\right\}$

(v) Transform the equation $x^3 - 6x^2 + 10x - 3 = 0$ in which co-efficient of x^2 term is zero.

Ans:- $x^3 - 2x + 1 = 0$

(vi) Transform the equation $x^4 + 8x^3 + x - 5 = 0$ so that the term containing the cubic power of 'x' is absent.

Ans:- $x^4 - 24x^2 + 65x - 55 = 0$

(vii) Find the polynomial equation whose roots are the translates of those of the equation $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ by 2.

Ans:- $x^4 - 9x^3 + 40x^2 - 80x + 80 = 0$

(viii) Solve the equation $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$.

Ans:- $\left(\frac{1}{3}, \frac{1}{2}, 2 \text{ and } 3\right)$

(ix) Solve the equation $4x^3 - 13x^2 - 13x + 4 = 0$.

Ans:- $\left\{-1, \frac{1}{4}, 4\right\}$

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Unit **5**

Permutations and Combinations

PERMUTATIONS

- **Fundamental principle :** If a work can be done in 'p' different ways and a second work can be done in 'q' different ways then the two works (one after the other) can be done in pq different ways.
- **Permutations :** Each arrangement that can be made by taking some or all of a number of things is called permutation.
- **Permutations when repetitions are not allowed:** If n, r are positive integers and $r \le n$, then the number of permutations of n distinct things taken r at a time is

$${}^{n}p_{r} = \text{n(n-1)(n-2)....... ....(n-r+1)} = \frac{n!}{(n-r)!}$$

- (i) The sum of the all r-digit numbers that can be formed using the given 'n' distinct non-zero digits $(1 \le r \le n \le 9)$ is

$$= {}^{n-1}\text{P}_{r-1} \times (\textit{Sum of the given digits}) \times (1.1.1........r\ digits)$$

  (ii) If '0' is one digit among the given 'n' digits, then we get that sum of the 'r' digit numbers that can be formed using the given 'n' digits (including '0') is

$$= {}^{n-1}\text{P}_{r-1} \times (\textit{Sum of the given digits}) \times (1.1.1........r\ times)$$

$$- {}^{n-2}\text{P}_{r-2} \times (\textit{Sum of the given digits}) \times (1.1.1........(r-1)\ times)$$

Theorem: **If n, r be natural numbers and $1 \le r \le n$ then ${}^{n}\mathbf{P}_{r} = {}^{n-1}\mathbf{P}_{r} + r.{}^{n-1}\mathbf{P}_{r-1}$.**

Proof

$${}^{n}\text{P}_{r} = \frac{n\ !}{(n-r)\ !}$$

$${}^{n-1}\text{P}_{r} + r.{}^{n-1}\text{P}_{r-1} = \frac{(n-1)!}{(n-r-1)!} + r.\frac{(n-1)!}{[(n-1)-(r-1)]!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r.\frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!(n-r)}{(n-r)(n-r-1)!} + \frac{r(n-1)!}{(n-r)!} = \frac{(n-1)!(n-r)}{(n-r)!} + \frac{r.(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r)!}\left[n-r+r\right] = \frac{(n-1)!.n}{(n-r)!} = \frac{n!}{(n-r)!} = {}^{n}\mathrm{P}_r$$

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1. If ${}^{n}\mathrm{P}_4 = 1680$, find 'n'.

Sol:- ${}^{n}\mathrm{P}_4 = 1680$

= 10 × 168

= 10 × 8 × 21

= 10 × 8 × 7 × 3

= 8 × 7 × 6 × 5

$\therefore n = 8$

2. If ${}^{n}P_3 = 1320$, find 'n'.

Sol:- ${}^{n}\mathrm{P}_3 = 1320$

= 10 × 132

= 12 × 11 × 10

$\therefore n = 12$

3. If ${}^{n+1}\mathrm{P}_5 : {}^{n}\mathrm{P}_6 = 2:7$, find 'n'.

Sol:- ${}^{n+1}\mathrm{P}_5 : {}^{n}\mathrm{P}_6 = 2:7$

$$\frac{(n+1)!}{(n-4)!} : \frac{n!}{(n-6)!} = 2:7$$

$$\frac{(n+1)n!}{(n-4)(n-5)(n-6)!} \times \frac{(n-6)!}{n!} = \frac{2}{7}$$

$$\frac{(n+1)}{(n-4)(n-5)} = \frac{2}{7}$$

$$7(n+1) = 2(n^2 - 9n + 20)$$

$$2n^2 - 25n + 33 = 0$$

$$(n-11)(2n-3) = 0$$

$$\therefore n = 11 \left(\therefore n \neq \frac{3}{2}\right)$$

4. Find n, if $^{n+1}P_5 : {}^nP_5 = 3:2$.

Sol:- $\frac{(n+1)!}{(n-4)!} \times \frac{(n-5)!}{n!} = \frac{3}{2}$

$\frac{(n+1)n!}{(n-4)(n-5)!} \times \frac{(n-5)!}{n!} = \frac{3}{2}$

$\frac{(n+1)}{(n-4)} = \frac{3}{2}$

$2n+2 = 3n-12$

$-n = -14 \qquad \therefore n = 14$

5. If $^{56}P_{(r+6)} : {}^{54}P_{(r+3)} = 30800:1$, find r.

Sol:- $^{56}P_{(r+6)} : {}^{54}P_{(r+3)} = 30800:1$

$\Rightarrow \frac{(56)!}{(56-(r+6))!} \times \frac{(54-(r+3))!}{(54)!} = \frac{30800}{1}$

$\Rightarrow \frac{(56)!}{(50-r)!} \times \frac{(51-r)!}{(54)!} = \frac{30800}{1}$

$\Rightarrow 56 \times 55 \times (51-r) = 30800$

$\Rightarrow (51-r) = \frac{30800}{56 \times 55} = 10$

$\Rightarrow r = 41 \cdot$

Problem for Practice

(i) If $^{12}P_5 + 5.{}^{12}P_4 = {}^{13}P_r$, find r Ans:- $r = 5$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

2. If $^nP_7 = 42 \ {}^nP_5$, then find n.

Sol:- $\frac{n!}{(n-7)!} = 42\frac{n!}{(n-5)!}$

$\frac{1}{(n-7)!} = 42\frac{1}{(n-5)(n-6)(n-7)!}$

$(n-5)(n-6) = 42$

$(n-5)(n-6) = 7 \times 6$

$(n-5)(n-6) = (12-5)(12-6)$

$\therefore n = 12$

2. Find the number of ways of permuting the letters of the word PICTURE so that

(i) all vowels come together

(ii) no two vowels come together

(iii) the relative positions of vowels and consonants are not disturbed.

Sol:- The word PICTURE has 3 vowels (I, U, E) and 4 consonants (P, C, T, R)

(i) Treat 3 vowels as one unit. Then we can arrange, 4 consonants + 1 unit of vowels in 5! ways. Now the 3 vowels among themselves can be permuted in 3! ways. Hence the number of permutations in which the 3 vowels come togother is

$5! \times 3! = 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1$

$= 720$

| I U E | | P | | C | | T | | R | $= 5\,! \times 3\,! = 720$

3! (under I U E); 5! (under all five units)

(ii) First arrange the 4 consonants in 4! ways. Then in between the vowels, in the beginning and in the ending, there are 5 gaps as shown below by the letter x.

× □ × □ × □ × □ ×

1 2 3 4 5

In these 5 places we can arrange the 3 vowels in 5_{P_3} ways. Thus the no. of words in which no two vowels come together $= 4! \times 5_{P_3} = 24 \times 60 = 1440$.

(iii) The three vowels can be arranged in their relative positions in 3! ways and 4 consonants can be arranged in their relative positions in 4! ways

| V | | C | | C | | V | | C | | V | | C |

No. of the required arrangements is 3! 4! = 144

3. Find the number of ways of arranging the letters of the word TRIANGLE so that the relative positions of the vowels and consonants are not disturbed.

Sol:- The word TRIANGLE has 3 vowels (A, E, I) and 5 consonants (T, R, N, G, L)

| C | | C | | V | | V | | C | | C | | C | | V |

The three vowels can be arranged in their relative positions in 3! ways. The five consonants can be arranged in their relative position in 5! ways

The no. of required arrangements $= 3! \times 5!$

$= (3 \times 2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1)$

$= (6)(120)$

$= 720$

4. Find the sum of all four digited numbers that can be formed using the digits 1, 2, 4, 5, 6 without repetition.

Sol:- n = 5, r = 4, digits = 1 , 2, 4, 5, 6

The sum of all 4-digits that can be formed using the digits 1, 2, 4, 5, 6 without repetition is

$= {}^{n-1}P_{r-1} \times$ (sum of the given digits) × 1111.......r times

$= {}^{4}P_{3} \times (1 + 2 + 4 + 5 + 6) \times 1111$

$= 24 \times 18 \times 1111 = 4,79,952$

5. Find the number of ways of arranging 6 boys and 6 girls in a row. In how many of these arrangements i) all the girls are together ii) no two girls are together iii) boys and girls come alternately.

Sol:- 6 boys and 6 girls are altogether 12 persons.

They can be arranged in a row in (12)! ways.

i) Treat the 6 girls as one unit. Then we have 6 boys and 1 unit of girls. They can be arranged in 7! ways. Now, the 6 girls can be arranged among themselves in 6! ways. Thus the number of ways in which all 6 girls are together is 7! x 6!.

ii) First arrange the 6 boys in a row in 6! ways. Then we can find 7 gaps between them (including the begining gap and ending gap) as shown below by the letter x :

× B × B × B × B × B × B ×

Thus we have 7 gaps and 6 girls. They can be arranged in ${}^{7}P_{6}$ ways. Hence, the number of arrangements in which no two girls sit together is $6! \times {}^{7}P_{6} = 7.6!.6!$.

iii) The row may begin with either a boy or a girl, that is, 2 ways. If it begins with a boy, then odd places will be occupied by boys and even places by girls. The 6 boys can be arranged in 6 odd places in 6! ways and 6 girls in the 6 even places in 6! ways. Thus the number of arrangemtns in which boys and girls come alternately is 2 × 6! × 6!.

6. Find the no. of ways of arranging 5 boys and 4 girls so that the row (i) begins with a boy and end with a girl (ii) begins and ends with boys.

Sol:- (i)

B								G

We fill the first place with one of the boys in 5 ways and last place with one of the girls in 4 ways.

The remaining 7 places can be filled with the remaining 7 persons (4 boys + 3 girls) in 7! ways.

Total no. of required arrangements $= 5 \times 4 \times 7\,!$

$= 20 \times 5040$

$= 1,00,800$

(ii) The total no. of persons is 9 (5 boys + 4 girls)

B								B

First we fill the first and last places with boys = ${}^5P_2$ ways

remaining 7 places with remaining 7 persons (3 boys + 4 girls) = 7! ways

Total number of required arrangements = 7! x ${}^5P_2$ = 1,00,800

7. Find the number of ways of arranging 5 different Mathematics books, 4 different Physics books and 3 different Chemistry books such that the books of the same subject are together.

Sol:- The number of ways of arranging Mathematics, Physics and Chemistry books are arranged in 3! ways.

5 different Mathmatics books are arranged themselves in 5! ways.

4 different Physics books are arranged themselves in 4! ways.

3 different Chemistry books are arranged themselves in 3! ways.

$\therefore$ The numbers of required arrangements = $3! \times 5! \times 4! \times 3!$

$= 6 \times 120 \times 24 \times 6 = 103680$.

Problems for Practice

(i) Find the number of all 4 letter words that can be formed using the letters of the word **"EQUATION"**. How many of these words begin with E? How many end with N? How many beging with E and end with N?

(Example 5.2.8 Text Book Page Number 156).

(ii) Find the sum of all 4 digited numbers that can be formed using the digits 0, 2, 4, 7, 8, without repetition.

(Excersie 5(a), Section II. Q. No 4. Text Book Page Number 167).

LONG ANSWER TYPE QUESTIONS (7 MARKS)

1. If the letters of the word MASTER are permuted in all possible ways and the words thus formed are arranged in the dictionary order, then find the rank of the words (i) REMAST (II) MASTER

Sol:- The letters of the given word in dictionary order is A, E, M, R, S, T

(i) REMAST

A __ __ __ __ __ $\rightarrow$ 5! ways

E __ __ __ __ __ $\rightarrow$ 5! ways

M __ __ __ __ __ → 5! ways

R A __ __ __ __ __ → 4! ways

R E A __ __ __ __ __ → 3! ways

R E M A S T → 1! ways

Rank of the word REMAST

$= 3 \times 5! + 4! + 3! + 1!$

$= 3(120) + 24 + 6 + 1 = 391$

(ii) MASTER

A __ __ __ __ __ → 5! ways

E __ __ __ __ __ → 5! ways

M A E __ __ __ → 3! ways

M A R __ __ __ → 3! ways

M A S E __ __ → 2! ways

M A S R __ __ → 2! ways

M A S T E R → 1! ways

Rank of the word MASTER

$= 2 \times 5! + 2 \times 3! + 2 \times 2! + 1$

$= 2(120) + 2(6) + 2(2) + 1$

$= 240 + 12 + 4 + 1$

$= 257$

2. If the letters of the word PRISON are permuted in all possible ways and the words thus formed are arranged in dictionary order, find the rank of the word PRISON.

Sol:- The letters of the given word in dictionary order is I, N, O, P, R, S

I __ __ __ __ __ → 5! ways

N __ __ __ __ __ → 5! ways

O __ __ __ __ __ → 5! ways

P I __ __ __ __ → 4! ways

P N __ __ __ __ → 4! ways

P O __ __ __ __ → 4! ways

P R I N __ __ → 2! ways

P R I O __ __ → 2! ways

P R I S N __ → 1! ways

P R I S O N → 1! ways

rank of the word PRISON

$= 3 \times 5! + 3 \times 4! + 2 \times 2! + 1! \times 2$

$= 360 + 72 + 4 + 2 = 438$

COMBINATIONS

Combination : A selection that can be formed by taking some or all of a finite set of things (or objects) is called a combination.

Example: The combinations formed by taking two things at a time from a set {A, B, C} are {A, B}, {A, C}, {B, C}

Observations :

- The number of combinations of 'n' dissimilar things taken r at a time is denoted by $^{n}C_{r}$ or $C(n,r)$ or $C\begin{pmatrix} n \\ r \end{pmatrix}$ or $\begin{pmatrix} n \\ r \end{pmatrix}$
- The number of combinations of 'n' distinct objects taken r at a time is $\frac{^{n}p_{r}}{r!}$

  i.e, $^{n}c_{r} = \frac{^{n}p_{r}}{r!} = \frac{n!}{(n-r)!\,r!}$

  $\therefore {}^{n}c_{r} = \frac{n!}{(n-r)!\,r!}$
- For any positive integer n, $^{n}c_{n} = {}^{n}c_{0} = 1,\ {}^{n}c_{1} = n$
- For $r, s \le n,\ if\ {}^{n}C_{r} = {}^{n}C_{s}$ then $r+s=n\ or\ r=s$
- $^{n}c_{r} = {}^{n}c_{n-r}$
- $^{n}c_{r-1} + {}^{n}c_{r} = {}^{n+1}c_{r}$

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1. If $^{n}C_{4} = {}^{n}C_{6}$ then find 'n'.

Sol:- $^{n}C_{4} = {}^{n}C_{6}$

$\Rightarrow n = 4+6 \quad \left(\because {}^{n}C_{r} = {}^{n}C_{s} \Rightarrow n = r+s\right)$

$\Rightarrow n = 10$

2. If 10 $^{n}C_{2} = 3 \cdot {}^{n+1}C_{3}$ then find 'n'.

Sol:- $10\frac{n!}{(n-2)!2!} = 3.\frac{(n+1)!}{(n+1-3)!3!}$

$10\frac{n!}{(n-2)!2!} = 3.\frac{(n+1)n!}{(n-2)!3!}$

$5 = \frac{3.(n+1)}{3\times2\times1} \qquad 10 = n+1 \qquad \therefore n = 9$

3. If $^{12}C_{r+1} = {}^{12}C_{3r-5}$ then find 'n'

Sol:- $^{12}C_{r+1} = {}^{12}C_{3r-5}$

If $^{n}C_{r} = {}^{n}C_{s}$ (or) $^{n}C_{r} = {}^{n}C_{s}$

$\Rightarrow n = r + s$ $\Rightarrow r = s$

$12 = r + 1 + 3r - 5$ $r + 1 = 3r - 5$

$12 = 4r - 4$ $-2r = -6$

$\Rightarrow 4r = 16$ $r = 3$

$\Rightarrow r = 4$

4. Find 'n' and r if $^{n}P_{r} = 1320$, $^{n}c_{r} = 220$

Sol:- $r! = \dfrac{^{n}P_{r}}{^{n}C_{r}} = \dfrac{1320}{220} = 6$

$r! = 3 \times 2 \times 1 = 3!$

$\therefore r = 3$

$^{n}P_{3} = 1320$

$^{n}P_{3} = 12 \times 11 \times 10$

$\therefore$ n = 12.

5. If $^{n}C_{5} = {}^{n}C_{6}$ then find $^{13}C_{n}$

Sol:- $^{n}C_{5} = {}^{n}C_{6}$

$\Rightarrow n = 5 + 6$

$\Rightarrow n = 11$

$\therefore {}^{13}C_{n} = {}^{13}C_{11} = {}^{13}C_{2} = \dfrac{13 \times 12}{2 \times 1} = 78$

6. Find the value of $^{10}C_{5} + 2.\,^{10}C_{4} + {}^{10}C_{3}$

Sol:- $^{10}C_{5} + 2.\,^{10}C_{4} + {}^{10}C_{3}$

$= \left(^{10}C_{5} + {}^{10}C_{4}\right) + \left(^{10}C_{4} + {}^{10}C_{3}\right)$

$= {}^{11}C_{5} + {}^{11}C_{4}$ $\left[\because {}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}\right]$

$= {}^{12}C_{5}$

$= \dfrac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 12 \times 11 \times 6 = 792$

7. If $^{n}P_{r} = 5040$ and $^{n}C_{r} = 210$ find 'n' and 'r'

Sol:- $r! = \frac{^{n}p_{r}}{^{n}C_{r}} = \frac{5040}{210} = 24$

$r! = 4\times3\times2\times1 = 4!$

$\therefore\ r = 4$

$^{n}P_{4} = 5040$

$^{n}P_{4} = 10\times9\times8\times7$

$\therefore n = 10$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

1. Prove that $^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}$

Sol:- $L.H.S. = {}^{n}C_{r-1} + {}^{n}C_{r}$

$$= \frac{n!}{(n-r+1)!(r-1)!} + \frac{n!}{(n-r)!r!}$$

$$= n!\left[\frac{1}{(n-r+1)!(r-1)!} + \frac{1}{(n-r)!r!}\right]$$

$$= n!\left[\frac{r}{(n-r+1)!(r-1)!\,r} + \frac{n-r+1}{(n-r+1)(n-r)!\,r!}\right]$$

$$= n!\left[\frac{r}{(n-r+1)!r!} + \frac{n-r+1}{(n-r+1)!\,r!}\right]$$

$$= n!\left[\frac{r+n-r+1}{(n-r+1)!r!}\right]$$

$$= \frac{n!\,(n+1)}{(n-r+1)!r!} = \frac{(n+1)!}{(n+1-r)!r!} = {}^{n+1}C_{r}$$

2. Find the no. of ways selecting 5 books from 9 different mathematics books such that a particular book is not included.

Sol:- No. of books = 9

Particular book is not included, so remaining no. of books = 8

The no. of selections required = $^{8}C_{5}$ ways

$$= \frac{8\times7\times6\times5\times4}{5\times4\times3\times2\times1} = 56$$

3. Find the no. of ways of selecting 3 vowels and 2 consonants from the letters of the word EQUATION.

Sol:- The word EQUATION has 5 vowels (E, O, U, A, I) and 3 consonants (Q, T, N)

Now, The no. of selecting 3 vowels from 5 vowels $= {}^5C_3$ ways

The no. of selecting 2 consonants from 3 consonants $= {}^3C_2$ ways

Total no. of selections $= {}^5C_3 \times {}^3C_2$ ways

$$= \frac{5\times4\times3}{3\times2\times1}\times\frac{3\times2}{2\times1} = 30$$

4. Find the number of 5 letter words can be formed using 3 consonants and 2 vowels from the letters of the word MIXTURE.

Sol:- The word MIXTURE has 3 vowels (I, U, E) and 4 consonants (M, X, T, R)

The no. of selecting 2 vowels from 3 vowels $= {}^3C_2$ ways

The no. of selecting 3 consonants from 4 consonants $= {}^4C_3$ ways

So, 5 letters (3 Consonants + 2 vowels) can be arranged in 5! ways

The no. of 5 letter words formed $= {}^4C_3 \times {}^3C_2 \times 5! = 1440$

LONG ANSWER TYPE QUESTIONS (7 MARKS)

1. Prove that $\mathbf{{}^{25}C_4 + \sum_{r=0}^{4} {}^{(29-r)}C_3 = {}^{30}C_4}$

Sol:- $L.H.S. = {}^{25}C_4 + \sum_{r=0}^{4} {}^{29-r}C_3$

$= {}^{25}C_4 + \left[{}^{29}C_3 + {}^{28}C_3 + {}^{27}C_3 + {}^{26}C_3 + {}^{25}C_3\right]$

$= \left({}^{25}C_4 + {}^{25}C_3\right) + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3$

$= \left({}^{26}C_4 + {}^{26}C_3\right) + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3$

$= \left({}^{27}C_4 + {}^{27}C_3\right) + {}^{28}C_3 + {}^{29}C_3$

$= \left({}^{28}C_4 + {}^{28}C_3\right) + {}^{29}C_3$

$= {}^{29}C_4 + {}^{29}C_3 = {}^{30}C_4 \quad \left[\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r\right]$

2. Simplify $\mathbf{{}^{34}C_5 + \sum_{r=0}^{4} {}^{38-r}C_4}$

Sol:- $^{34}C_5 + \sum_{r=0}^{4} {}^{38-r}C_4$

$= {}^{34}C_5 + \left[{}^{38}C_4 + {}^{37}C_4 + {}^{36}C_4 + {}^{35}C_4 + {}^{34}C_4\right]$

$= \left({}^{34}C_5 + {}^{34}C_4\right) + {}^{35}C_4 + {}^{36}C_4 + {}^{37}C_4 + {}^{38}C_4$

$= \left({}^{35}C_5 + {}^{35}C_4\right) + {}^{36}C_4 + {}^{37}C_4 + {}^{38}C_4$ $\left[\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r\right]$

$= \left({}^{36}C_5 + {}^{36}C_4\right) + {}^{37}C_4 + {}^{38}C_4$

$= \left({}^{37}C_5 + {}^{37}C_4\right) + {}^{38}C_4$

$= {}^{38}C_5 + {}^{38}C_4$

$= {}^{39}C_5$

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Unit **6**

Binomial Theorem

➪ Binomial Theorem for positive Integral Index

If x and a are real numbers, then

$(x+a)^n = n_{C_0} x^n a^0 + n_{C_1} x^{n-1}.\ a^1 + n_{C_2} x^{n-2} a^2 + ----- + n_{C_r}\ x^{n-r}\ a^r + --- + n_{C_n} x^0 a^n$ for all $n \in N$

➪ Some Important Conclusions from the Binomial Theorem

1. $(x+a)^n = \sum_{r=0}^{n} n_{C_r}\ x^{n-r}\ a^r \quad ------*$

   Since r can have values from 0 to n, therefore the total number of terms in the expansion of $(x+a)^n$ is n+1

2. Replacing 'a' by '-a' in $(*)$, we get

   $(x-a)^n = \sum_{r=0}^{n} (-1)^r.\ n_{C_r}\ x^{n-r}\ a^r$

   $= n_{C_0}\ x^n\ a^0 - n_{C_1}\ x^{n-1} a^1 + n_{C_2} x^{n-2} a^2 ---- + (-1)^n\ n_{C_n}\ x^0\ a^n$

   Thus, the terms in the expansion of $(x-a)^n$ are alternatively positive and negative. The last term is positive or negative according as n is even or odd.

3. Putting $x = 1$ and $a = x$ in $(*)$, we get $(1+x)^n = \sum_{r=0}^{n} n_{C_r}\ x^r = \sum_{r=0}^{n} C_r\ x^r$

   $= C_0 + C_1 x + C_2 x^2 + ----- + C_n\ x^n$

   This is called Standard binomial expansion.

4. The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is n_{C_r} or C_r

5. **General term :-**

   The $(r+1)^{th}$ term in the expansion of $(x+a)^n$ is

   $T_{r+1} = n_{C_r}\ x^{n-r}\ a^r$

6. The number of terms in the trinomial expansion of $(a+b+c)^n$ is $\left(\frac{(n+1)(n+2)}{2}\right)$

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1. Expand $(4x+5y)^7$ using binomial theorem.

Sol:- $(4x+5y)^7 = \sum_{r=0}^{n} n_{C_r}\ X^{n-r}\ a^r$, Where $X = 4x,\ a = 5y$ and $n = 7$

$= \sum_{r=0}^{7} 7C_r (4x)^{7-r}(5y)^r$

2. Expand $\left(\frac{2}{3}x+\frac{7}{4}y\right)^5$ using binomial theorem.

Sol:- $\left(\frac{2x}{3}+\frac{7y}{4}\right)^5 = \sum_{r=0}^{n} n_{C_r}\ X^{n-r}\ a^r$, Where $X = \frac{2x}{3},\ a = \frac{7y}{4}$ and $n = 5$

$= \sum_{r=0}^{5} 5C_r \left(\frac{2x}{3}\right)^{5-r}\left(\frac{7y}{4}\right)^r$

3. Find the 6^{th} term in the expansion of $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$.

Sol:- comparing $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$ with $(x+a)^n$, we have

$X = \frac{2x}{3},\ a = \frac{3y}{2}$ and $n = 9$

The general term is $T_{r+1} = n_{C_r}\ X^{n-r}\ a^r = 9C_r\left(\frac{2x}{3}\right)^{9-r}\left(\frac{3y}{2}\right)^r$

$= 9C_r .\ 2^{9-2r}\ 3^{2r-9}.\ x^{9-r}\ y^r$

Putting $r = 5$, we get $T_6 = 9C_5\ 2^{-1}.3^1x^4y^5 = \frac{9.8.7.6.5}{1.2.3.4.5}.\frac{1}{2}.3.x^4\ y^5$

$= 189\,x^4\,y^5$

4. Find the 7^{th} term in the expansion of $\left(\frac{4}{x^3}+\frac{x^2}{2}\right)^{14}$.

Sol:- Comparing $\left(\frac{4}{x^3}+\frac{x^2}{2}\right)^{14}$ with $(X+a)^n$, we have

$X = \frac{4}{x^3},\ a = \frac{x^2}{2}$ and $n = 14$

The general term is $T_{r+1} = n_{C_r}\ X^{n-r}\ a^r = 14C_r\left(\frac{4}{x^3}\right)^{14-r}\left(\frac{x^2}{2}\right)^r$

$= 14C_r\ 2^{28-3r}.\ x^{5r-42}$

Putting r=6, we get $T_7 = 14C_6\ 2^{10}\ x^{-12} = \frac{14C_6 \times 2^{10}}{x^{12}}$

5. Find the 3^{rd} term from the end in the expansion of $\left(x^{\frac{-2}{3}} - \frac{3}{x^2}\right)^8$.

Sol:- Comparing $\left(x^{\frac{-2}{3}} - \frac{3}{x^2}\right)^8$ with $(X+a)^n$, we have

$X = x^{\frac{-2}{3}}$, $a = \frac{-3}{x^2}$ and $n = 8$

3rd term from the end = $n - k + 2 = 8 - 3 + 2 = 7^{th}$ term from the beginning [$\because k = 3$]

Now, $T_{r+1} = n_{C_r}\ X^{n-r}\ a^r = 8C_r \left(x^{\frac{-2}{3}}\right)^{8-r} \left(\frac{-3}{x^2}\right)^r$

Putting r = 6, we get $T_7 = 8C_6\left(x^{\frac{-2}{3}}\right)^2 \left(\frac{-3}{x^2}\right)^6 = 8C_2\ 3^6\ x^{\frac{-4}{3}-12} = \frac{8.7}{1.2} \times 3^6.x^{\frac{-40}{3}} = \frac{28 \times 3^6}{x^{40/3}}$

6. Find the number of terms in the expansion of $(2x + 3y + z)^7$.

Sol:- Here $n = 7$

Therefore, the number of terms in the expansion of $(2x + 3y + z)^7$ is

$= \frac{(n+1)(n+2)}{2} = \frac{(7+1)(7+2)}{2} = \frac{8 \times 9}{2} = 36$

7. Find the coefficient of x^{-6} in the expansion of $\left(3x - \frac{4}{x}\right)^{10}$.

Sol:- $\left(3x - \frac{4}{x}\right)^{10}$

$T_{r+1} = {}^{10}C_r \,.\, (3x)^{10-r} \,.\, \left(\frac{-4}{x}\right)^r$

$= {}^{10}C_r \,.\, (3)^{10-r} \,.\, (-4)^r \,.\, x^{10-2r}$

Therefore, the coefficient of $x^{-6} = x^r$ in the expansion of $\left(3x - \frac{4}{x}\right)^{10}$

$\Rightarrow 10 - 2r = -6$

$\Rightarrow 2r = 16 \Rightarrow r = 8.$

Coefficient of x^{-6} is

$= {}^{10}C_8 \,.\, 3^{10-8} \,.\, (-4)^8$

${}^{10}C_8 \,.\, 3^2 \,.\, 4^8 = \frac{10.9}{1.2} \,.\, 3^2.4^8 = 405 \times 4^8$

8. Find the coefficient of x^{11} in the expansion of $\left(2x^2 + \frac{3}{x^3}\right)^{13}$.

Sol:- $\left(2x^2 + \frac{3}{x^3}\right)^{13}$

$T_{r+1} = {}^{13}C_r \,.\, \left(2x^2\right)^{13-r} \,.\, \left(\frac{3}{x^3}\right)^r$

$T_{r+1} = {}^{13}C_r \,.\, (2)^{13-r} \,.\, (3)^r \,.\, x^{26-5r}$

Therefore, the coefficient of $x^{11} = x^r$ in the expansion of $\left(2x^2 + \frac{3}{x^3}\right)^{13}$

$\Rightarrow 26 - 5r = 11 \Rightarrow 5r = 15 \Rightarrow r = 3.$

Coefficient of x^{11} is

$${}^{13}C_3\ 2^{13-3}\ 3^3 = \frac{13.12.11}{1.2.3}.\ 2^{10}\ .\ 3^3 = 286 \times 2^{10}.3^3$$

9. Find the term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$.

Sol:- $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$

$$T_{r+1} = {}^{10}C_r \,.\left(\sqrt{\frac{x}{3}}\right)^{10-r} .\left(\frac{3}{2x^2}\right)^r$$

$$T_{r+1} = {}^{10}C_r \,.\, \frac{(3)^{\frac{3r}{2}-5}}{2^r} .\, (x)^{5-\frac{5r}{2}} \qquad \text{------(1)}$$

Therefore, to find the term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$

put $5 - \frac{5r}{2} = 0$

$\Rightarrow 5r = 10 \Rightarrow r = 2.$

From eq. (1), independent term $= {}^{10}C_2\,\frac{3^{-2}}{2^2} = \frac{10.9}{1.2}.\frac{1}{3^2.2^2} = \frac{5}{4}.$

10. Find the middle term(s) in the expansion of $\left(\frac{3x}{7} - 2y\right)^{10}$.

Sol:- Comparing $\left(\frac{3x}{7} - 2y\right)^{10}$ with $(x+a)^n$, we have

$x = \frac{3x}{7}$, $a = -2y$ and $n = 10$

Since $n = 10$ is even therefore $T_{\frac{n}{2}+1}$ i.e., T_6 is the middle term in the given binomial expansion

Now, $T_{r+1} = n_{C_r}\ x^{n-r}\ a^r = 10_{c_r}\left(\frac{3x}{7}\right)^{10-r}(-2y)^r$

Putting r = 5, we get $T_6 = 10_{C_5}\left(\frac{3x}{7}\right)^5(-2y)^5 = -10_{C_5}\left(\frac{3}{7}\right)^5\ 2^5\ x^5\ y^5$

11. Find the middle term(s) in the expansion of $\left(4a + \frac{3b}{2}\right)^{11}$.

Sol:- Comparing $\left(4a + \frac{3b}{2}\right)^{11}$ with $(x+A)^n$, we have

$x = 4a,\ A = \frac{3b}{2}$ and $n = 11$

Since n = 11 is odd, therefore $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$ i.e., T_6 and T_7 are middle terms in the expansion of $\left(4a + \frac{3b}{2}\right)^{11}$

Now, $T_{r+1} = {}^nC_r\, x^{n-r}\ A^r = {}^{11}C_r\,(4a)^{11-r}\left(\frac{3b}{2}\right)^r$

Putting r = 5 and r = 6, we get

$T_6 = {}^{11}C_5\,(4a)^6 . \left(\frac{3b}{2}\right)^5$ and $T_7 = {}^{11}C_6 (4a)^5\left(\frac{3b}{2}\right)^6$

$T_6 = 77.2^8.3^6.a^6\,b^5$ and $T_7 = 77.2^5.\,3^7.\,a^5\,b^6$

$\left(\because\ {}^{11}C_5 = {}^{11}C_6 = \frac{11.10.9.8.7}{1.2.3.4.5} = 77\times3\times2\right)$

LONG ANSWER TYPE QUESTIONS (7 MARKS)

1. Find the numerically greatest term(s) in the expansion of $(4a - 6b)^{13}$ when a = 3 and b = 5.

Sol:- Write $(4a - 6b)^{13} = \left[4a\left(1 - \frac{6b}{4a}\right)\right]^{13} = (4a)^{13}\left(1 - \frac{3b}{2a}\right)^{13}$

First we find the numerically greatest term in the expansion of $\left(1 - \frac{3b}{2a}\right)^{13}$

Comparing $\left(1 - \frac{3b}{2a}\right)^{13}$ with $(1 + x)^n$, we have

$x = \frac{-3b}{2a} = \frac{-3\times5}{2\times3} = \frac{-5}{2}$ and $n = 13$

Now, $m = \frac{(n+1)\,|\,x\,|}{1+|\,x\,|} = \frac{(13+1)\left|\frac{-5}{2}\right|}{1+\left|\frac{-5}{2}\right|} = \frac{14\times\frac{5}{2}}{\frac{7}{2}} = \frac{70}{7} = 10$

$\therefore\ T_m$ and T_{m+1} i.e., T_{10} and T_{11} are numerically greatest terms in the expansion of $\left(1 - \frac{3b}{2a}\right)^{13}$

and $|T_{10}| = |T_{11}|$

$T_{10} = {}^{13}C_9 . x^9 = {}^{13}C_9\left(\frac{-5}{2}\right)^9 = \frac{-{}^{13}C_9\,5^9}{2^9}$

$T_{11} = {}^{13}C_{10}\ x^{10} = {}^{13}C_{10}\left(\frac{-5}{2}\right)^{10} = \frac{{}^{13}C_{10}\,5^{10}}{2^{10}}$

Hence, the numerically greatest terms in the expansion of $(4a-6b)^{13}$ are T_{10} and T_{11}. They are

$$T_{10} = (4\times3)^{13} \times \frac{-{}^{13}C_9.5^9}{2^9} = -{}^{13}C_9.2^{17}.3^{13}.5^9 = -143\times2^{17}.3^{13}.5^{10} \quad \text{and}$$

$$T_{11} = (4\times3)^{13} \times \frac{{}^{13}C_{10}.5^{10}}{2^{10}} = {}^{13}C_{10}.2^{16}.3^{13}.5^{10} = 143\times2^{16}.3^{13}.5^{10}$$

2. Find the numerically greatest term(s) in the expansion of $(2+3x)^{10}$ when $x=\frac{11}{8}$.

Sol:- Write $(2+3x)^{10} = \left[2\left(1+\frac{3x}{2}\right)\right]^{10} = 2^{10}\left(1+\frac{3x}{2}\right)^{10}$

First we find the numerically greatest term in the expansion of $\left(1+\frac{3x}{2}\right)^{10}$

Comparing $\left(1+\frac{3x}{2}\right)^{10}$ with $(1+x)^n$, we have

$$x=\frac{3x}{2} = \frac{3}{2}\left(\frac{11}{8}\right) = \frac{33}{16} \text{ and } n=10$$

Now, $m=\frac{(n+1)|x|}{1+|x|} = \frac{(10+1)\left|\frac{33}{16}\right|}{1+\left|\frac{33}{16}\right|} = \frac{\frac{363}{16}}{\frac{49}{16}} = \frac{363}{49}$ is not an integer

and $[m] = \left[\frac{363}{49}\right] = 7$

Therefore, $T_{[m]+1} = T_8$ is the numerically greatest term in the expansion of $\left(1+\frac{3x}{2}\right)^{10}$ and

$$T_8 = {}^{10}C_7\ x^7 = 10_{C_7}\left(\frac{33}{16}\right)^7$$

Hence, the numerically greatest term in the expansion of $(2+3x)^{10}$ is

$$T_8 = 2^{10}.{}^{10}C_7\left(\frac{33}{16}\right)^7 = \frac{{}^{10}C_7(33)^7}{2^{18}}$$

3. Find the numerically greatest term(s) in the expansion of $(3x-4y)^{14}$ when $x=8$, and $y=3$.

Sol:- Write $(3x-4y)^{14} = \left[3x\left(1-\frac{4y}{3x}\right)\right]^{14} = (3x)^{14}\left(1-\frac{4y}{3x}\right)^{14}$

First we find the numerically greatest term in the expansion of $\left(1-\frac{4y}{3x}\right)^{14}$

Comparing $\left(1-\frac{4y}{3x}\right)^{14}$ with $(1+x)^n$, we have

$$x=\frac{-4y}{3x} = \frac{-4\times3}{3\times8} = \frac{-1}{2} \text{ and } n=14$$

Now, $m=\frac{(n+1)|x|}{1+|x|} = \frac{(14+1)\left|\frac{-1}{2}\right|}{1+\left|\frac{-1}{2}\right|} = \frac{\frac{15}{2}}{\frac{3}{2}} = 5$ is an integer

Therefore, T_m and T_{m+1} i.e., T_5 and T_6 are numerically greatest terms in the expansion of $\left(1-\frac{4y}{3x}\right)^{14}$ and $|T_5| = |T_6|$

$$T_5 = {}^{14}C_4\ x^4 = {}^{14}C_4\left(\frac{-1}{2}\right)^4 = \frac{{}^{14}C_4}{2^4} \quad \text{and}$$

$$T_6 = {}^{14}C_5\ x^5 = {}^{14}C_5\left(\frac{-1}{2}\right)^5 = \frac{-{}^{14}C_5}{2^5}$$

Hence, the numerically greatest term in the expansion of $(3x-4y)^{14}$ are T_5 and T_6. They are

$$T_5 = \frac{(3x)^{14}.{}^{14}C_4}{2^4} = \frac{(24)^{14}.{}^{14}C_4}{2^4} = 1001 \times 2^{38} \times 3^{14} \text{ and} \qquad [\because x = 8]$$

$$T_6 = \frac{(3x)^{14}(-{}^{14}C_5)}{2^5} = \frac{-(24)^{14}.{}^{14}C_5}{2^5} = -1001 \times 2^{37} \times 3^{14} \qquad [\because x = 8]$$

4. Find the numerically greatest term(s) in the expansion of $(4+3x)^{15}$ when $x = \frac{7}{2}$.

Sol:- $(4+3x)^{15} = 4^{15}\left(1+\frac{3x}{4}\right)^{15} = (1+X)^n$ where $n = 15,\ X = \frac{3x}{4} = \frac{3}{4}\times\frac{7}{2} = \frac{21}{8}$.

Now, $m = \frac{(n+1)|X|}{|X|+1} = \frac{(15+1)(21/8)}{21/8+1} = \frac{16\times 21}{29} = \frac{336}{29} = 11.59.$

$[m] = [11.59] = 12.$

$\therefore$ 12th term is the numerically greatest term and

$$T_{12} = {}^{15}C_{11}\ (4)^{15-11}(3x)^{11} = {}^{15}C_4\, 2^8\, 21^{11} / 2^{11} = {}^{15}C_4\ 21^{11} / 2^3 \qquad \left[\because x = \frac{7}{2}\right]$$

Problems for Practice

Find the numerically greatest term(s) in the expansion of

(i) $(3x+5y)^{12}$ when $x = \frac{1}{2},\ y = \frac{4}{3}$ Ans:- $T_{11} = 12_{C_{10}}\left(\frac{3}{2}\right)^2\left(\frac{20}{3}\right)^{10}$

(ii) $(3+7x)^n$ when $x = \frac{4}{5},\ n = 15$ Ans:- $T_{11} = 15_{C_{11}}\left(\frac{28}{5}\right)^{11}\ 3^4$

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Unit **7**

Partial Fractions

➪ **Rational Fraction :** If $f(x)$ and $g(x)$ are two polynomials and $g(x)$ is a non-zero polynomial, then $\frac{f(x)}{g(x)}$ is called a rational fraction.

➪ **Proper and Improper Fractions :** A rational fraction $\frac{f(x)}{g(x)}$ is called a proper fraction, if the degree of $f(x)$ is less than the degree of $g(x)$. Otherwise it is called an improper fraction.

➪ **Rule (1) :** Let $\frac{f(x)}{g(x)}$ be a proper fraction. To each non repeated factor $(ax+b)$ of $g(x)$, there will be a partial fraction of the form $\frac{A}{ax+b}$, where 'A' is a non-zero real number, to be determined.

➪ **Rule (2) :-** Let $\frac{f(x)}{g(x)}$ be a proper fraction. To each non repeated factor $(ax+b)^n$, $a \neq 0$, where 'n' is a positive integer, of $g(x)$ there will partial fraction of the form, $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \text{..........................} + \frac{A_n}{(ax+b)^n}$, where $A_1, A_2, \text{................} A_n$ are to be determined constants.

➪ **Rule (3) :-** Let $\frac{f(x)}{g(x)}$ be a proper fraction. To each non repeated quadratic factor $\left(ax^2+bx+c\right)$, $a \neq 0$ of $g(x)$ there will be a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where $A, B \in R$

➪ **Rule (4) :-** Let $\frac{f(x)}{g(x)}$ be a proper fraction. If $n(>1) \in N$ is the largest exponent so that $\left(ax^2+bx+c\right)^n$, $a \neq 0$, is a factor of $g(x)$, then corresponding to each such factor, there will be partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2} + \text{..........................} + \frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$$

where $A_1, A_2, \text{...........} A_n$, $B_1, B_2, \text{..................} B_n \in R$ to be determined.

➪ If $\frac{f(x)}{g(x)}$ is an improper fraction with degree of $f(x) >$ the deg. of $g(x)$ then by using division algorithm, $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$, $where\left(g(x) \neq 0\right)$, $\frac{r(x)}{g(x)}$ is a proper fraction. Further $\frac{r(x)}{g(x)}$ can be resolved into partial fractions using the above rules.

SHORT ANSWER QUESTIONS (4 MARKS)

1. Resolve $\frac{5x+1}{(x+2)(x-1)}$ into partial fractions.

Sol:- Let $\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

Where A and B are non-zero real numbers to be determined.

Then $\frac{5x+1}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

$\therefore\ A(x-1)+B(x+2) = 5x+1$ .... (1)

Puting $x = 1$ in eq. (1), we get

3B = 5+1, i.e. B = 2

Puting $x = -2$ in eq. (1), we get

–3A = –9, i.e. A = 3

$\therefore\ \frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$

2. Resolve $\frac{2x+3}{(x+1)(x-3)}$ into partial fractions.

Sol:- Let $\frac{2x+3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$\Rightarrow A = \frac{2(-1)+3}{-1-3} = -\frac{1}{4},\ B = \frac{2(3)+3}{3+1} = \frac{9}{4}$

$\Rightarrow \frac{2x+3}{(x+1)(x-3)} = \frac{-1}{4(x+1)} + \frac{9}{4(x-3)}$

3. Resolve $\frac{x^2+5x+7}{(x-3)^3}$ into partial fractions.

Sol:- Let $\frac{x^2+5x+7}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$

Where A, B and C are constants to be determined.

$$\therefore\ \frac{x^2+5x+7}{(x-3)^3}=\frac{A(x-3)^2+B(x-3)+C}{(x-3)^3}$$

$$\therefore\ x^2+5x+7=Ax^2+(B-6A)x+(9A-3B+C) \qquad \text{...... (1)}$$

Now, comparing the coefficients of like powers of x in (1), we get

A = 1, B – 6A = 5, 9A – 3B + C = 7

Solving these equations we get A = 1, B = 11, C = 31.

$$\therefore\ \frac{x^2+5x+7}{(x-3)^3}=\frac{1}{(x-3)}+\frac{11}{(x-3)^2}+\frac{31}{(x-3)^3}$$

4. Resolve $\mathbf{\dfrac{x^2+13x+15}{(2x+3)(x+3)^2}}$ into partial fractions.

Sol:- Here $(2x+3)$ is a linear factor and $(x+3)$ is repeated linear factor. We apply Rules I and II and write

$$\frac{x^2+13x+15}{(2x+3)(x+3)^2}=\frac{A}{2x+3}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$$

Where A, B and C are constants to be determined.

$$\therefore\ A(x+3)^2+B(2x+3)(x+3)+C(2x+3)=x^2+13x+15 \qquad \text{...... (1)}$$

Putting $x=-3$ in (1) we get, –3C = –15 or C = 5

Putting $x=\frac{-3}{2}$ in (1), we get, $\frac{9A}{4}=\frac{-9}{4}$ or A = –1.

Now comparing the coefficients in eq. (1), we get

A + 2B = 1

i.e. –1 +2B = 1 $(\because\ A=-1)$

B = 1

$$\therefore\ \frac{x^2+13x+15}{(2x+3)(x+3)^2}=\frac{-1}{2x+3}+\frac{1}{x+3}+\frac{5}{(x+3)^2}$$

5. Resolve $\mathbf{\dfrac{x+4}{(x^2-4)(x+1)}}$ into partial fractions.

Sol:- $$\frac{x+4}{(x^2-4)(x+1)}=\frac{x+4}{(x-2)(x+2)(x+1)}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x-2}$$

$$x+4=A(x+2)(x-2)+B(x+1)(x-2)+c(x+1)(x+2) \qquad \text{........(1)}$$

Putting $x=-1$ in (1), we get

$-1+4=A(1-4)+0+0\Rightarrow -3A=3\Rightarrow A=-1$

Putting $x=-2$ in (1), we get

$$2 = B(-2+1)(-2-2) \Rightarrow 4B = 2 \Rightarrow B = \frac{2}{4} = \frac{1}{2}$$

Putting $x = 2$ in (1), we get

$$6 = C(2+1)(2+2) \Rightarrow 12C = 6 \Rightarrow C = \frac{6}{12} = \frac{1}{2}$$

$$A = -1, B = \frac{1}{2}, C = \frac{1}{2}$$

$$\therefore \frac{x+4}{(x^2-4)(x+1)} = \frac{-1}{x+1} + \frac{1}{2(x+2)} + \frac{1}{2(x-2)}$$

6. Resolve $\frac{\mathbf{x^2 - 3}}{\mathbf{(x+2)(x^2+1)}}$ into partial fractions.

Sol:- Let $\frac{x^2-3}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$

$$x^2 - 3 = A(x^2+1) + (Bx+C)(x+2)$$

$$x = -2 \Rightarrow 4 - 3 = A(4+1) + 0 \Rightarrow 5A = 1 \Rightarrow A = \frac{1}{5}$$

Comparing x^2 coefficients

$$1 = A + B \Rightarrow B = 1 - A = 1 - \frac{1}{5} = \frac{4}{5}$$

Comparing Constants

$$-3 = A + 2C \Rightarrow 2C = -3 - A = -3 - \frac{1}{5} = \frac{-16}{5} \Rightarrow 2C = \frac{-16}{5}$$

$$\Rightarrow C = \frac{-16}{5x2} = \frac{-8}{5}$$

$$\therefore \frac{x^2-3}{(x+2)(x^2+1)} = \frac{1}{5(x+2)} + \frac{4x-8}{5(x^2+1)}$$

7. Resolve $\frac{\mathbf{2x^2+1}}{\mathbf{x^3 - 1}}$ into partial fractions.

Sol:- $\frac{2x^2+1}{x^3-1} = \frac{2x^2+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$$2x^2 + 1 = A(x^2+x+1) + (Bx+C)(x-1)$$

$$= Ax^2 + Ax + A) + Bx^2 - Bx + Cx - C$$

$$2x^2 + 1 = \left(A+B\right)x^2 + \left(A-B+C\right)x + \left(A-C\right)$$

Comparing x^2 coefficients $A+B=2$ ....(1)

Comparing x coefficients $A-B+C=0$ ....(2)

Comparing Constants $A-C=1$ ....(3)

(1) + (2)

$A+B=2$

$A-B+C=0$

$2A+C=2$ ....(4)

$A-C=1$

$3A=3 \Rightarrow A=1$

$(1) \Rightarrow 1+B=2 \Rightarrow B=2-1=1$

$(3) \Rightarrow 1-C=1 \Rightarrow -C=1-1=0 \Rightarrow C=0$

$\therefore A=1,\ B=1,\ C=0$

$$\therefore \frac{2x^2+1}{x^3-1}=\frac{1}{x-1}+\frac{x}{x^2+x+1}$$

8. Resolve $\frac{3x^3-2x^2-1}{x^4+x^2+1}$ into partial fractions.

Sol:- $x^4+x^2+1=\left(x^2+1\right)^2-x^2=\left(x^2+x+1\right)\left(x^2-x+1\right)$

$$\therefore Let\ \frac{3x^3-2x^2-1}{x^4+x^2+1}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$$

$3x^3-2x^2-1=(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$

comparing the coefficients of $x^3,\ x^2,\ x$ and constants in (1), we get

$A+C=3$ ....(2)

$-A+B+C+D=-2$ ....(3)

$A-B+C+D=0$ ....(4)

$B+D=-1$ ....(5)

$(2) \Rightarrow C=3-A$ ....(6)

$(5) \Rightarrow D=-1-B$ ....(7)

Putting these values in (3), we get

$-A+B+3-A-1-B=-2 \Rightarrow -2A=-4 \Rightarrow A=\frac{-4}{-2}=2$ $\therefore A=2$

Putting these values in (4), we get

$A-B+3-A-1-B=0$

$-2B=-2 \Rightarrow B=\frac{-2}{-2}=1$ $\therefore B=1$

$(6) \Rightarrow C = 3 - 2 = 1 \qquad \therefore C = 1$

$(7) \Rightarrow D = -1 - 1 = -2 \qquad \therefore D = -2$

$$\therefore \frac{3x^3 - 2x^2 - 1}{x^4 + x^2 + 1} = \frac{2x+1}{x^2+x+1} + \frac{x-2}{x^2-x+1}$$

9. Resolve $\mathbf{\dfrac{x^4}{(x-1)(x-2)}}$ into partial fractions.

Sol:- $$\frac{x^4}{(x-1)(x-2)} = \frac{x^4}{(x^2-3x+2)}$$

$$= x^2 + 3x + 7 + \frac{15x-14}{x^2-3x+2} \text{ ....(1)}$$

$$= \frac{15x-14}{x^2-3x+2} = \frac{15x-14}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$15x - 14 = A(x-2) + B(x-1)$

$x = 1 \Rightarrow 15 - 14 = A(1-2) + B(1-1)$

$1 = A(-1) + B(0)$

$1 = -A \Rightarrow A = -1$

$x = 2 \Rightarrow 30 - 14 = A(2-2) + B(2-1)$

$16 = A(0) + B(1)$

$16 = B$

$\therefore A = -1,\ B = 16$

$$\therefore \text{ From (1) } \frac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 - \frac{1}{x-1} + \frac{16}{x-2}$$

10. Resolve $\mathbf{\dfrac{x^3}{(x-a)(x-b)(x-c)}}$ into partial fractions.

Sol:- $$\frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$x^3 = (x-a)(x-b)(x-c) + A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$$

$$x = a \Rightarrow a^3 = 0 + A(a-b)(a-c) + 0 + 0 \Rightarrow A = \frac{a^3}{(a-b)(a-c)}$$

$$x = b \Rightarrow b^3 = 0 + 0 + B(b-a)(b-c) + 0 \Rightarrow B = \frac{b^3}{(b-a)(b-c)}$$

$$x = c \Rightarrow c^3 = 0 + 0 + 0 + C(c-a)(c-b) \Rightarrow C = \frac{c^3}{(c-a)(c-b)}$$

$$\therefore \frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{a^3}{(a-b)(a-c)(x-a)} + \frac{b^3}{(b-a)(b-c)(x-b)} + \frac{c^3}{(c-a)(c-b)(x-c)}$$

Problem for Practice

(i) Resolve $\frac{5x+6}{(2+x)(1-x)}$ into partial fractions.

Ans: $\frac{5x+6}{(2+x)(1-x)} = \frac{-4}{3(2+x)} + \frac{11}{3(1-x)}$

(ii) Resolve $\frac{2x+3}{(x-1)^3}$ into partial fractions.

Ans: $\frac{2x+3}{(x-1)^3} = \frac{2}{(x-1)^2} + \frac{5}{(x-1)^3}$

(iii) Resolve $\frac{x^2-2x+6}{(x-2)^3}$ into partial fractions.

Ans: $\frac{x^2-2x+6}{(x-2)^3} = \frac{1}{(x-2)} + \frac{2}{(x-2)^3}$

(iv) Resolve $\frac{1}{(x-1)^2(x-2)}$ into partial fractions.

Ans: Example 3, Page No. 266 from Text Book.

(v) Resolve $\frac{9}{(x-1)(x+2)^2}$ into partial fractions.

Ans: $\frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$

(vi) Resolve $\frac{2x^2+3x+4}{(x-1)(x^2+2)}$ into partial fractions.

Ans: $\frac{2x^2+3x+4}{(x-1)(x^2+2)} = \frac{3}{x-1} + \frac{2-x}{x^2+2}$

(vii) Resolve $\frac{x^2}{(x-1)(x-2)}$ into partial fractions.

Ans: $\frac{x^2}{(x-1)(x-2)} = 1 - \frac{1}{x-1} + \frac{4}{x-2}$

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Unit **8**

Measures of Dispersion

⇨ **Mean deviation for ungrouped data :**

(i) Mean deviation about the mean $= \frac{1}{n}\left(\sum x_i - \overline{x}\right)$

($n \rightarrow$ number of observations, $\overline{x} \rightarrow$ mean, $x_i \rightarrow$ observations)

(ii) Mean deviation about median $= \frac{1}{n}\sum |x_i - \text{med}|$

($x_i \rightarrow$ observations, $n \rightarrow$ number of observations, med. $\rightarrow$ median)

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1. Find the mean deviation from the mean of the following discrete data.
6, 7, 10, 12, 13, 4, 12, 16

Sol:- The Arithmetic mean of the given data is

$$\overline{x} = \frac{6+7+10+12+13+4+12+16}{8} = 10$$

The absolute values of the deviations :

$\left|x_i - \overline{x}\right|$ are 4, 3, 0, 2, 3, 6, 2, 6

The mean deviation from the mean

$$= \frac{\sum_{i=1}^{8}\left|x_i - \overline{x}\right|}{8} = \frac{4+3+0+2+3+6+2+6}{8} = \frac{26}{8} = 3.25$$

2. Find the mean deviation from the median of the following discrete data.
6, 7, 10, 12, 13, 4, 12, 16

Sol:- Expressing the data points in the ascending order of magnitude, we get

4, 6, 7, 10, 12, 12, 13, 16

Then the median of these eight observations

$$b = \frac{10+12}{2} = 11$$

Then the absolute values

$|x_i - b|$ are 7, 5, 4, 1, 1, 1, 2, 5

The mean deviation from the median

$$= \frac{\sum_{i=1}^{8} |x_i - b|}{8} = \frac{26}{8} = 3.25 .$$

3. Find the mean deviation about the mean for the following data

3, 6, 10, 4, 9, 10

Sol:- Mean of the given data is

$$\bar{x} = \frac{3+6+10+4+9+10}{6} = \frac{42}{6} = 7$$

The absolute values of the deviataions are 4, 1, 3, 3, 2, 3

The required mean deviation about mean is

$$= \frac{\sum_{i=1}^{6} |x_i - \bar{x}|}{6} = \frac{4+1+3+3+2+3}{6} = \frac{16}{6} = 2.67$$

4. Find the mean deviation about the median for the followind data

4, 6, 9, 3, 10, 13, 2

Sol:- The ascending order of the observations in the given data : 2, 3, 4, 6, 9, 10, 13

Median of the given data is

$$M = \frac{7+1}{2} \text{th term} = 6$$

The absolute values of the deviataions are 4, 3, 2, 0, 3, 4, 7.

The required mean deviation about median is

$$\text{M.D.} = \frac{\sum_{i=1}^{7} |x_i - M|}{7} = \frac{4+3+2+0+3+4+7}{7} = \frac{23}{7} = 3.29$$

Problems for Practice

(i) Find the mean deviation about the mean for the following data

38, 70, 48, 40, 42, 55, 63, 46, 54, 44

(ii) Find the mean deviation about the median for the following data

13, 17, 16, 11, 13, 10, 16, 11, 18, 12, 17

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Unit **9**

Probability

Probability: If a random experiment results in 'n' exhaustive, mutually exclusive and equally likely elementary events and m of them are favourable to happening of an event E, then the probability of occurance of E denoted by P(E) is defined by $P(E) = \frac{m}{n}$.

SHORT AND LONG ANSWER TYPE QUESTIONS

1. A fair coin is tossed 200 times. Find the probability of getting a head an odd number of times.

Sol:- Experiments : A fair coin is tossed 200 times

$n = 2 \times 2 \times .... \times 2$ (200 times)

$= 2^{200}$

E : The event of getting a head an odd number of times

$$m = {}^{200}C_1 + {}^{200}C_3 + {}^{200}C_5 + ... + {}^{200}C_{199} = 2^{200-1} = 2^{199}$$

$$P(E) = \frac{m}{n} = \frac{2^{199}}{2^{200}} = \frac{1}{2}$$

2. Out of 30 consecutive integers, two integers are drawn at random. Find the probability that their sum is odd.

Sol:- **Experiment :** Drawing two integers from 30 consecutive integers

$$n = {}^{30}C_2 = \frac{30 \times 29}{2} = 15 \times 29$$

E : The Event that the sum of the two integers drawn is odd

$m = {}^{15}C_1 \times {}^{15}C_1$ [sum of two integers is odd if out of two one is even and one is odd. There are 15 even and 15 odd integers in 30 consecutive integers]

$= 15 \times 15$

$= 225$

$$P(E) = \frac{m}{n} = \frac{225}{15 \times 29} = \frac{15}{29}$$

3. Find the probability of throwing a total score of 7 with 2 dice ?

Sol:- Experiment : Throwing 2 dice

n = 6 x 6 = 36 $S = \{(1,1), (1,2), ..., (1,6)$

$(2,1), (2,2), ..., (2,6)$

....

....

$(6,1), (6,2), ..., (6,6)\}$

E : The event of getting a total of 7

$E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$m = n(E) = 6$

$P(E) = \frac{m}{n} = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

4. A page is opened at random from a book containg 200 pages. What is the probability that the number on the page is a perfect square ?

Sol:- Experiment : Opening a page from a book of 200 pages

$n = 200_{C_1} = 200$

E : The event that the opened page number is a perfect square

$E = \{1^2, 2^2, 3^2, ...14^2\} = \{1, 4, 9, ...1\ 96\}$

$m = n(E) = 14$

$P(E) = \frac{m}{n} = \frac{n(E)}{n(S)} = \frac{14}{200} = \frac{7}{100}$

5. If 4 fair coins are tossed simultaneously, then find the probability that 2 heads and 2 tails appear.

Sol:- Experiment : Tossing 4 fair coins

$n = n(s) = 2^4 = 16$

E : The event of getting 2 heads and 2 tails

E = { HHTT, HTHT, THTH, HTTH, THHT, TTHH }

$m = n(E) = 6 = \frac{4!}{2!2!}$

$P(E) = \frac{m}{n} = \frac{n(E)}{n(S)} = \frac{6}{16} = \frac{3}{8}$

6. Find the probability that a non-leap year contains (i) 53 Sundays (ii) 52 Sundays only.

Sol:- A non leap year contains 365 days, i.e. 52 Sundays and 1 day

(i) E : The event that a non leap year contains 53 Sundays

$P(E_1)$ = probability that the 1 day is Sunday

$= \frac{1}{7}$ [favourable case is only 1 that is Sunday and total no. of days of a week is 7]

(ii) E_2 : The event that a non leap year contains only 52 Sundays

$P(E_2)$ = The probability that the 1 day is not a Sunday

= The probability that the 1 day is MON, TUE, WED, THUR, FRI, SAT

$= \frac{6}{7}$ [Favourable cases are 6 and total no. of days of a week is 7]

Probability - Axiomatic approach

Let S be the Sample space of a random experiment which is finite. Then a function $P : P(S) \rightarrow R$ satisfying the following axioms is called a Probability Function [P(S) is the power set of S]

(i) $P(E) \geq 0 \ \forall E \in P(S)$ [axiom of Non-negativity]

(ii) P(S) = 1

(iii) If $E_1, E_2 \in P(S)$ and $E_1 \cap E_2 = \phi$ then

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$ [axiom of additivity]

Note :- Let S be a sample space of a random experiment and P be a probability function on P(S), then

(i) $P(\phi) = 0$

(ii) $P(E^C)=1-P(E)$

(iii) If $E_1 \subseteq E_2$, then $P(E_2 - E_1) = P(E_2) - P(E_1)$

(iv) If $E_1 \subseteq E_2$, then $P(E_1) \leq P(E_2)$

Addition theorem on probability

7. State and prove addition theorem on probability.

Statement :- If E_1, E_2 are any two events of a random experiment and P is a probability function then $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

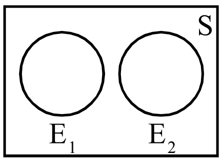
Proof:- **Case(i)** Suppose $E_1 \cap E_2 = \phi$

$\Rightarrow P(E_1 \cap E_2) = 0$ --------(1)

We know that $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ [from axiom of additivity]

$\Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2) - 0$

$\Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ [$\because$ from (1)]



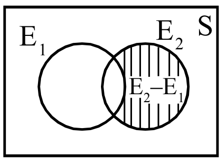
Case (ii) :-

Suppose $E_1 \cap E_2 \neq \phi$

$E_1 \cup E_2 = E_1 \cup (E_2 - E_1)$

$\Rightarrow P(E_1 \cup E_2) = P(E_1 \cup (E_2 - E_1))$

$\Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2 - E_1)$ [$\because E_1 \cap (E_2 - E_1) = \phi$]



$\Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2 - (E_1 \cap E_2)) \quad [\because E_2 - E_1 = E_2 - (E_2 \cap E_1)]$

$\Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$[\because P(A - B) = P(A) - P(B)$ if $B \subseteq A$ and since $E_1 \cap E_2 \subseteq E_2]$

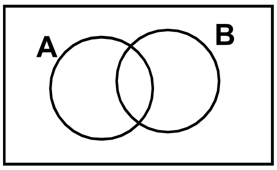
No.	Event	Set-Theoretic description
1.	Event A or Event B to occur	$A \cup B$
2.	Both event A and B to occur	$A \cap B$
3.	Neither A nor B occurs	$A^C \cap B^C = (A \cup B)^C$
4.	A occurs but B does not occur	$A \cap B^C$
5.	Exactly one of the events A, B to occur	$(A \cap B^C) \cup (A^C \cap B)$ (or) $(A - B) \cup (B - A)$ (or) $(A \cup B) - (A \cap B)$

Applications of Addition Theorem on Probability

8. If A and B are two events, then show that i) P(A – B) = P(A) – P(A $\cap$ B).

ii) The probability that exactly one of them occurs, is given by

P(A) + P(B) – 2P(A $\cap$ B).



Sol:- i) $A = (A - B) \cup (A \cap B)$ and $\varnothing = (A - B) \cap (A \cap B)$

$\therefore P(A) = P[(A - B) \cup (A \cap B)] = P(A - B) + P(A \cap B)$

$\Rightarrow P(A - B) = P(A) - P(A \cap B).$

ii) The probability that exactly one of them occurs

$= P[(A - B) \cup (B - A)] = P(A - B) + P(B - A)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B).$

9. Suppose A and B are events with P(A) = 0.5, P(B) = 0.4 and P(A $\cap$ B) = 0.3. Find the probability that i) A does not occur ii) neither A nor B occurs.

Sol:- i) The pobability that A does not occur = $P(A^c) = 1 - P(A) = 1 - 0.5 = 0.5$.

ii) The probability that neighter A nor B occurs = $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)] = 1 - [0.5 + 0.4 - 0.3] = 0.4.$

10. In an Experiment of drawing a card at random from a pack, the event of getting a spade is denoted by A and getting a pictured card (King, Queen or Jack) is denoted by B. Find the probabilities of A, B, $A \cap B$ and $A \cup B$.

Sol:- Experiment : Drawing a card from a pack

$n = n(S) = {}^{52}C_1 = 52$

A : The event of getting a spade card

$n(A) = {}^{13}C_1 = 13, \quad P(A) = \frac{n(A)}{n(S)} = \frac{13}{52}$

B : The event of getting a pictured card

$n(B) = {}^{12}C_1$ [$\because$ 4 kings + 4 Queens + 4 Jacks]

$= 12$

$$P(B) = \frac{n(B)}{n(S)} = \frac{12}{52}$$

$A \cap B$: The event that the drawn card is a spade pictured card

$n(A \cap B) = {}^{3}C_1 = 3$ [$\because$ one spade king + one spade Queen + one spade Jack]

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{3}{52}$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ($\because$ Addition Theorem)

$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{13 + 12 - 3}{52} = \frac{22}{52}$$

$$P(A \cup B) = \frac{11}{26}$$

11. In a class of 60 boys and 20 girls, half of the boys and half of the girls know cricket. Find the probability of the event that a person selected from the class is either a boy, or a girl knowing cricket.

Sol:- Total no. of persons in the class = 80 ($\because$ 60B+20G)

Experiment : Selecting a person from the class

$n = n(S) = {}^{80}C_1 = 80$

A : The event that the selected person is a boy

B : The event that the selected person is a girl knowing cricket.

$A \cap B = \phi$

$A \cup B$: The event that the selected person is a boy or a girl knowing cricket.

$n(A) = {}^{60}C_1 = 60$; $n(B) = {}^{10}C_1 = 10$ ($\because$ Half of 20 Girls)

$$P(A) = \frac{n(A)}{n(S)} = \frac{60}{80}, \quad P(B) = \frac{n(B)}{n(S)} = \frac{10}{80}$$

$P(A \cup B) = P(A) + P(B)$ $[\because A \cap B = \phi]$

$$= \frac{60}{80} + \frac{10}{80} = \frac{70}{80} = \frac{7}{8}$$

12. If one ticket is randomly selected from tickets numbered from 1 to 30, then find the probability that the number on the ticket is (i) a multiple of 5 or 7 (ii) multiple of 3 or 5.

Sol:- Experiment : Selecting one ticket from 30 tickets numbered from 1 to 30

$n = n(S) = {}^{30}C_1 = 30$

A : The event that the number on the ticket is a multiple of 5

B : The event that the number on the ticket is a multiple of 7

C : The event that the number on the ticket is a multiple of 3

A = { 5, 10, 15, 20, 25, 30} B = {7, 14, 21, 28} C = {3, 6, 9, 12, 15, 18, 21, 24, 27, 30}

$n(A) = 6,\ n(B) = 4,\ n(C) = 10$

$A \cap B =$ The event that the number is a multiple of both 5 and 7

$A \cap C =$ The event that the number is a multiple of both 5 and 3

$A \cap B = \phi, \quad A \cap C = \{15, 30\}$

$n(A \cap C) = 2$

(i) $P(A \cup B) = P(A) + P(B)\ [\because A \cap B = \phi]$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = \frac{6}{30} + \frac{4}{30} = \frac{10}{30} = \frac{1}{3}$$

(ii) $P(A \cup C) = P(A) + P(C) - P(A \cap C)$

$$= \frac{n(A)}{n(S)} + \frac{n(C)}{n(S)} - \frac{n(A \cap C)}{n(S)} = \frac{6}{30} + \frac{10}{30} - \frac{2}{30} = \frac{14}{30} = \frac{7}{15}$$

13. A, B, C are 3 news papers from a city. 20% of the population read A, 16% read B, 14% read C, 8% both A and B, 5% both A and C, 4% both B and C and 2% all the three. Find the percentage of the population who read atleast one news paper.

Sol:- $A \cup B \cup C$: The event that read atleast one of A or B or C

Given $P(A) = \frac{20}{100},\ P(B) = \frac{16}{100},\ P(C) = \frac{14}{100},\ P(A \cap B) = \frac{8}{100}$

$P(A \cap C) = \frac{5}{100},\ P(B \cap C) = \frac{4}{100},\ P(A \cap B \cap C) = \frac{2}{100}$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{4}{100} - \frac{5}{100} + \frac{2}{100} = \frac{52 - 17}{100}$$

$$= \frac{35}{100}$$

$\therefore$ 35% population read atleast one of A, B, C

14. The probability for a contractor to get a road contract is 2/3 and to get a building contract is 5/9. The probability to get atleast one contract is 4/5. Find the probability that he gets both the contracts.

Sol:- Let A be the event of getting road contract, B be the event of getting building contract.

$\therefore P(A) = 2/3, P(B) = 5/9, P(A \cup B) = 4/5$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{2}{3} + \frac{5}{9} - \frac{4}{5} = \frac{30 + 25 - 36}{45} = \frac{19}{45}.$$

15. In a committee of 25 members, each member is proficient either in Mathematics or in Statistics or in both. If 19 of these are proficient in Mathematics, 16 in Statistics, find the probability that a person selected from the committee is proficient in both.

Sol:- When a person is chosen at random from a committee of 25 members, let *A* be the event that the person is an expert in Mathematics, *B* be the event that the person is an expert in Statistics and S be the sample sapce. Since 19 members are experts in Mathematics and 16 members are experts in Statics,

$$\Rightarrow P(A \cup B) = P(S) = P(A) + P(B) - P(A \cap B) = 1 \Rightarrow \frac{19}{25} + \frac{16}{25} - P(A \cap B) = 1$$

$$\Rightarrow P(A \cap B) = \frac{19}{25} + \frac{16}{25} - 1 = \frac{19+16-25}{25} = \frac{10}{25} = \frac{2}{5}.$$

16. A, B, C are three horses in a race. The probability of A to win the race is twice of B, and the probability of B is twice that of C. What are the probabilities of A, B, C to win the race?

Sol:- Let, A, B, C be the events that the horses A, B, C win the race respectively.

Given P(A) = 2P(B), P(B) = 2P(C).

$\therefore$ P(A) = 2P(B) = 2 × 2P(C) = 4P(C)

Since the horses A, B and C run the race, $A \cup B \cup C = S$ and A, B, C are mutually disjoint.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) = 4P(C) + 2P(C) + P(C) = 1 \quad [\because P(S) = 1]$$

$$\Rightarrow 7P(C) = 1 \Rightarrow P(C) = \frac{1}{7}.$$

$$P(A) = 4P(C) = \frac{4}{7}; \quad P(B) = 2P(C) = \frac{2}{7}.$$

Conditional probability

Conditional Event :- Suppose A and B are two events of a random experiment. If the event 'B' occurs after the occurance of the Event 'A', then the event : "happening of B after the happening of A" is called Conditional Event and is denoted by B/A. Similarly A/B stands for the event : "happening of 'A' after the happening of B"

Conditional probability :- If 'A' and 'B' are two events of a sample space S and P(A) ≠ 0, then the probability of B after the occurance of 'A', is called the conditional probability of B given A and is denoted by $P(B/A)$.

It is defined as

$$P(B/A) = \frac{P(B \cap A)}{P(A)} \quad [\text{Where } P(A) \neq 0]$$

Similarly $P(A/B) = \frac{P(A \cap B)}{P(B)} \quad [\text{where } P(B) \neq 0]$

Multiplication theorem of probability

17. State and prove multiplication theorem of probability.

Statement :- If A and B are two events of a random experiment with $P(A) \neq 0$ and $P(B) \neq 0$ then $P(A \cap B) = P(A)P(B/A) = P(B)P(A/B)$

Proof:- By the definition of conditional probability

$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$

$$\Rightarrow P(B \cap A) = P(A).P(B/A)............(1)$$

Similarly $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$\Rightarrow P(A \cap B) = P(B).P(A/B)............(2)$$

From (1) and (2)

$$P(A \cap B) = P(A)P(B/A) = P(B)P(A/B)$$

Note : $P(A \cap B \cap C) = P(A)P(B/A)P(C/A \cap B)$

Def : Independent events :- Two events A and B of an experiment are called independent if $P(A \cap B) = P(A).P(B)$

Note :-

(i) If A and B are independent then $P(B/A) = P(B)$. That is conditional probability of B given **A** is same as the probability of B (That is probability of B does not depend on A)

(ii) If A and B are independent, then $P(A/B) = P(A)$ (A does not depend on B)

(iii) IF A, B and C are independent $P(A \cap B \cap C) = P(A).P(B).P(C)$

18. A pair of dice is thrown. Find the probability that either of the dice shows 2 when their sum is 6

Sol:- Experiment : Throwing a pair of dice

n = n(s) = 6x6 = 36

A : The event that the sum of the two number on the dice is 6

B : The event that 2 appears on either of the dice

Required event B/A : The event that either of the dice shows 2 when their sum is 6

A = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}

B = {(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2)}

$A \cap B = \{(2, 4), (4, 2)\}$

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{36}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{36}$$

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{2/36}{5/36} = 2/5$$

19. An urn contains 7 red and 3 black balls. Two balls are drawn without replacement. What is the probability that the second ball is red if it is known that the first ball drawn is red.

Sol:- Total no. of balls in the urn = 10 ($\because$ 7R+3B)

E_1 : The event that the first ball drawn is red

E_2 : The event that the second ball drawn is red

The required event : E_2/E_1 the second ball drawn is red if first ball drawn is Red

$n(E_2/E_1) = 6_{C_1} = 6$ ($\because$ There are only 6 Red balls since the first ball drawn is red and it is not replaced)

$n(S) = 9_{C_1}$ ($\because$ There are only 9 balls since one ball is already drawn)

$$P(E_2/E_1) = \frac{n(E_2/E_1)}{n(S)} = \frac{6}{9} = \frac{2}{3}$$

Applications of Multiplication theorem

20. A box contains 4 defective and 6 good bulbs. Two bulbs are drawn at random without replacement. Find the probability that both the bulbs drawn are good.

Sol:- A : The event that the first bulb drawn is good

B : The event that the second bulb drawn is good.

$n(A) = 6_{C_1} = 6$

$n(B/A) = 5_{C_1} = 5$ ($\because$ One good bulb is alread drawn), $P(A) = \frac{6}{10}$, $P(B/A) = \frac{5}{9}$

Required event $A \cap B$ **:** The event that both the bulbs drawn are good.

$P(A \cap B) = P(A).P(B/A)$

$$= \frac{6}{10}.\frac{5}{9} = \frac{1}{3}$$

21. A bag contains 10 identical balls of which 4 are blue and 6 are red three balls are taken out at random from the bag one after other. Find the probability that all the three balls drawn are red.

Sol:- A : The event that the first ball drawn is red

B : The event that the second ball drawn is red

C : The event that the third ball drawn is red

Required event : $A \cap B \cap C$: the event that all the three balls drawn are red.

$n(A) = {}^6C_1 = 6, \quad n\left(\frac{B}{A}\right) = {}^5C_1 = 5$ ($\because$ One red ball is already drawn)

$n(C/A \cap B) = {}^4C_1 = 4$ ($\because$ two red balls drawn already)

$$P(A \cap B \cap C) = P(A).P(B/A).P(C/A \cap B)$$

$$= \frac{6}{10}.\frac{5}{9}.\frac{4}{8}$$

$$= \frac{1}{6}$$

22. Bag B_1 contains 4 white and 2 black balls. Bag B_2 contains 3 white and 4 black balls. A bag is drawn at random and a ball is chosen at random from it. What is the probability that the ball drawn is white.

Sol:- Let E_1 : event of choosing bag B_1

E_2 : event of choosing bag B_2

Then $P(E_1) = P(E_2) = \frac{1}{2}$

Let W : event that the ball chosen from the selected bag is white

Then, $P(W/E_1) = \frac{4}{6} = \frac{2}{3}$, $P(W/E_2) = \frac{3}{7}$

Required event $A : (W \cap E_1) \cup (W \cap E_2)$ ($\because$ Selecting a bag and then selecting a white ball from it. That is selecting first bag and a white ball from it or selecting second bag and a white ball from it)

$P(A) = P\left[(E_1 \cap W) \cup (E_2 \cap W)\right]$

$= P(E_1 \cap W) + P(E_2 \cap W) \left[\because (W \cap E_1) \cap (W \cap E_2) = \phi\right]$

$= P(E_1).P(W/E_1) + P(E_2).P(W/E_2)$

$= \frac{1}{2}.\frac{2}{3} + \frac{1}{2}.\frac{3}{7}$

$= \frac{1}{3} + \frac{3}{14}$

$= \frac{23}{42}$

23. There are 3 black and 4 white balls in one bag, 4 black and 3 white balls in the second bag. A die is rolled and the first bag is selected if the die shows up 1 or 3, and the second bag for the rest. Find the probability of drawing a black ball, from the bag thus selected.

Sol:- E_1 : Event of selecting first bag

E_2 : Event of selecting second bag

$P(E_1) = \frac{2}{6} = \frac{1}{3}$ ($\because$ 1 or 3 out of six outcomes)

$P(E_2) = \frac{4}{6} = \frac{2}{3}$ ($\because$ Remaining 2,4,5,6 out of 6)

B : Event that the ball drawn is black from the selected bag.

$P(B/E_1) = \frac{3}{7}, \quad P(B/E_2) = \frac{4}{7}$

Required event $W = (E_1 \cap B) \cup (E_2 \cap B)$

$P(W) = P\left[(E_1 \cap B) \cup (E_2 \cap B)\right]$

$= P(E_1 \cap B) + P(E_2 \cap B) \quad \left[\because (E_1 \cap B) \cap (E_2 \cap B) = \phi\right]$

$= P(E_1).P(B/E_1) + P(E_2).P(B/E_2)$

$= \frac{1}{3}.\frac{3}{7} + \frac{2}{3}.\frac{4}{7}$

$= \frac{3+8}{21} = \frac{11}{21}$

Applications of Independent events

24. A and B are independent events and P(A) = 0.2, P(B)=0.5 Then Find the values of

(i) $P(A/B)$ **(ii)** $P(B/A)$ **(iii)** $P(A \cap B)$ **(iv)** $P(A \cup B)$

Sol:- (i) $P(A/B) = P(A)$ ($\because$ A, B are independent)

$= 0.2$

(ii) $P(B/A) = P(B)$ ($\because$ A, B are independent)

$= 0.5$

(iii) $P(A \cap B) = P(A).P(B)$ ($\because$ A, B are independent)

$= (0.2)(0.5)$

$= 0.1$

(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.2 + 0.5 - 0.1$

$= 0.6$

25. A speaks truth in 75% of the cases and B in 80% cases. What is the probability that their statements about an incident do not match.

Sol:- Let E_1 : The event that A speaks truth about an incident

E_2 : The event that B speaks truth about an incident

Then $P(E_1) = \frac{75}{100} = \frac{3}{4}$; $P(E_2) = \frac{80}{100} = \frac{4}{5}$

$\Rightarrow P(E_1^C) = 1 - \frac{3}{4} = \frac{1}{4}$; $P(E_2^C) = 1 - \frac{4}{5} = \frac{1}{5}$

Required event E : The event that their statements do not match about the incident

$E = \left(E_1 \cap E_2^C\right) \cup \left(E_1^C \cap E_2\right)$

$P(E) = P\left[\left(E_1 \cap E_2^C\right) \cup \left(E_1^C \cap E_2\right)\right]$

$= P\left(E_1 \cap E_2^C\right) + P\left(E_1^C \cap E_2\right)$ $\left[\because \left(E_1 \cap E_2^C\right) \cap \left(E_1^C \cap E_2\right) = \phi\right]$

$P(E) = P(E_1).P(E_2^C) + P(E_1^C).P(E_2)$ $\left[\because E_1, E_2 \text{ are independent}\right]$

$= \left(\frac{3}{4}\right)\left(\frac{1}{5}\right) + \left(\frac{1}{4}\right)\left(\frac{4}{5}\right)$

$= \frac{3+4}{20} = \frac{7}{20}$

26. A problem in calculus is given to two students A and B whose chances of solving it are 1/3 and 1/4 respectively. Find the probability of the problem being solved if both of them try independently.

Sol:- Let E_1 : The event that the problem is solved by A

E_2 : The event that the problem is solved by B

Given $P(E_1) = \frac{1}{3}$ $P(E_2) = \frac{1}{4}$

Required Event : $E_1 \cup E_2$: The event that the problem being solved (that is the problem is solved by either A or B or by both)

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{1}{3} + \frac{1}{4} - P(E_1).P(E_2) \quad [\because E_1\ E_2 \text{ are independent}]$$

$$= \frac{1}{3} + \frac{1}{4} - \left(\frac{1}{3}\right).\left(\frac{1}{4}\right)$$

$$= \frac{4+3-1}{12} = \frac{6}{12} = \frac{1}{2}$$

27. If A and B are independent events with P(A) = 0.6, P(B) = 0.7 then find

(i) $P(A \cap B)$ **(ii)** $P(A \cup B)$ **(iii)** $P(B/A)$ **(iv)** $P(A^C \cap B^C)$

Sol:- Given A, B are independent events.

i) $P(A \cap B) = P(A)\,P(B) = 0.6 \times 0.7 = 0.42$

ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.7 - 0.42 = 0.88$

iii) $P(B/A) = P(B) = 0.7.$

iv) $P(A^C \cap B^C) = P(A^C) . P(B^C)$

$= [1 - P(A)]\,[1 - P(B)]$ $[\because A^C, B^C$ are also independent events$]$

$= 0.4 \times 0.3 = 0.12.$

28. If A, B, C are three independent events such that $P(A \cap B^c \cap C^c) = 1/4$, $P(A^c \cap B \cap C^c) = 1/8$, $P(A^c \cap B^c \cap C^c) = 1/4$, then find P(A), P(B), P(C).

Sol:- Let $P(A) = x, P(B) = y, P(C) = z$

$$P(A \cap B^c \cap C^c) = 1/4 \Rightarrow P(A)P(B^c)P(C^C) = 1/4 \Rightarrow x(1-y)(1-z) = 1/4 \rightarrow (1)$$

$$P(A^c \cap B \cap C^c) = 1/8 \Rightarrow P(A^c)P(B)P(C^C) = 1/8 \Rightarrow (1-x)y(1-z) = 1/8 \rightarrow (2)$$

$$P(A^c \cap B^c \cap C^c) = 1/4 \Rightarrow P(A^c)P(B^c)P(C^c) = 1/4 \Rightarrow (1-x)(1-y)(1-z) = 1/4 \to (3)$$

$$\frac{(1)}{(3)} \Rightarrow \frac{x(1-y)(1-z)}{(1-x)(1-y)(1-z)} = \frac{1/4}{1/4} \Rightarrow \frac{x}{1-x} = 1 \Rightarrow x = 1-x \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}.$$

For $x = \frac{1}{2}$, $\quad \frac{(1)}{(2)} \Rightarrow \frac{\frac{1}{2}(1-y)(1-z)}{(1-\frac{1}{2})y(1-z)} = \frac{1/4}{1/8} \Rightarrow \frac{1-y}{y} = 2 \Rightarrow 1-y = 2y$

$$\Rightarrow 3y = 1 \Rightarrow y = \frac{1}{3}.$$

For $x = \frac{1}{2}$, $\quad (1) \Rightarrow \frac{1}{2}\left(1-\frac{1}{3}\right)(1-z) = \frac{1}{4} \Rightarrow \frac{1}{2} \times \frac{2}{3}(1-z) = \frac{1}{4}$

$$\Rightarrow 1-z = \frac{3}{4} \Rightarrow z = \frac{1}{4}.$$

$$\therefore P(A) = \frac{1}{2}, P(B) = \frac{1}{3}, P(C) = \frac{1}{4}.$$

29. If A, B are two events with $P(A \cup B) = 0.65$, $P(A \cap B) = 0.15$, then the find value of $P(A^c) + P(B^c)$.

Sol:- $P(A^c) + P(B^c) = 1 - P(A) + 1 - P(B) = 2 - [P(A) + P(B)]$

$= 2 - [P(A \cup B) + P(A \cap B)] = 2 - [0.65 + 0.15] = 1.2$

30. A fair die is rolled. Consider the events A = {1, 3, 5} , B = {2, 3} and C = {2, 3, 4, 5}. Find

i) $P(A \cap B)$, $P(A \cup B)$ **ii) $P(A/B)$, $P(B/A)$**

iii) $P(A/C)$, $P(C/A)$ **iv) $P(B/C)$, $P(C/B)$**

Sol:- Let S be the sample space. Then S = {1, 2, 3, 4, 5, 6}.

Given A = {1, 3, 5}, B = {2, 3}, C = {2, 3, 4, 5}

$(A \cap B) = \{3\}$, $(A \cup B) = \{1, 2, 3, 5\}$, $A \cap C = \{3, 5\}$, $B \cap C = \{2, 3\}$

i) $P(A \cap B) = 1/6$, $P(A \cup B) = 4/6 = 2/3$.

ii) $P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{2}$, $\quad P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{3}$

iii) $P(A/C) = \frac{n(A \cap C)}{n(C)} = \frac{2}{4} = \frac{1}{2}$, $\quad P(C/A) = \frac{n(A \cap C)}{n(A)} = \frac{2}{3}$

Iv) $P(B/C) = \frac{n(B \cap C)}{n(C)} = \frac{2}{4} = \frac{1}{2}$, $\quad P(C/B) = \frac{n(B \cap C)}{n(B)} = \frac{2}{2} = 1.$

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Unit **10**

Random Variables & Probability Distributions

➩ **Random Variable :-** Suppose 'S' is a sample space of random experiment, then any function $X : S \to R$ is called a 'Random Variable.'

➩ **Probability Distribution Function :-** Let 'S' be a Sample Space and $X : S \to R$ be a random variable. The function $F : R \to R$ defined by $F(x) = P(X \le x)$, is called "Probability distribution function" of the random variable 'x'.

➩ **Discrete Random variable :-** A random variable X whose range is either finite or countably infinite is called a discrete random variable. If the range of X is

$\{x_1, x_2, ...x_n\}$ *or* $\{x_1, x_2, ...\}$,

then X is called "Discrete Random Variable".

If $X : S \to R$ is a discrete random variable with range $\{x_1, x_2, x_3..........\}$, then

(i) $\mathrm{P}(\mathrm{X} = x_i) \ge 0$ for every i.

(ii) $\sum_{i \ge 1} \mathrm{P}(\mathrm{X} = x_i) = 1$

➩ **Mean :** $\mu = \Sigma x_i \, \mathrm{P}(\mathrm{X} = x_i)$

➩ **Variance :** $\sigma^2 = \Sigma x_i^2 \, \mathrm{P}(\mathrm{X} = x_i) - \mu^2$

Binomial Distribution :

$$\mathrm{P}(\mathrm{X} = x) = {}^nC_x \, p^x q^{n-x}, \; x \in \{0, 1, 2................n\}$$

(i) 'n' and 'p' are called 'parameters' of binomial distribution.

(ii) Mean of Binomial Distribution $(\mu) = np$

(iii) Variance of Binomial Distribution $(\sigma^2) = npq$

Poisson Distribution :

$$\mathrm{P}(\mathrm{X} = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \; x \in \{0, 1, 2, ...\} \text{ and } \lambda > 0$$

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1. If the mean and variance of a binomial variable X are 2.4 and 1.44 respectively, find $P(1 < X \leq 4)$

Sol:- Mean $= np = 2.4$ ....(i)

Variance $= npq = 1.44$ ....(ii)

Dividing (2) by (1)

$$\frac{npq}{np} = \frac{1.44}{2.4}$$

$$q = \frac{144}{240} = \frac{3}{5} = 0.6$$

But p+q=1

P+0.6=1 $\Rightarrow$ P=1-0.6=0.4

Substituting P=0.4 in (1)

n(0.4)=2.4

$$\Rightarrow n = \frac{2.4}{0.4} = \frac{24}{4} = 6$$

$$P(1 < X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4)$$

$$= {}^6C_2\, q^4 p^2 + {}^6C_3\, q^3 p^3 + {}^6C_4\, q^2 p^4$$

$$= {}^6C_2 \left(\frac{3}{5}\right)^4 \left(\frac{2}{5}\right)^2 + {}^6C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^3 + {}^6C_4 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^4$$

$$= 15.\frac{3^2.3^2}{5^4}.\frac{2^2}{5^2} + 20.\frac{3^2.3^1}{5^3}.\frac{2^2.2^1}{5^3} + 15.\frac{3^2}{5^2}.\frac{2^2.2^2}{5^4}$$

$$= (15.9)\left(\frac{6^2}{5^6}\right) + (20.6)\left(\frac{6^2}{5^6}\right) + (15.4)\left(\frac{6^2}{5^6}\right)$$

$$= \left(\frac{6^2}{5^6}\right)[135 + 120 + 60]$$

$$= \frac{36 \times 315}{15625} = \frac{2268}{3125}$$

2. A poisson variable satisfies P(X=1)=P(X=2), Find P(X=5) ?

Sol:- $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!},\ (\lambda > 0)$

Given that P(X=1)=P(X=2)

$\Rightarrow \dfrac{e^{-\lambda}\lambda^1}{1!} = \dfrac{e^{-\lambda}\lambda^2}{2!}$, Cross multiplying we get

$\Rightarrow 2\lambda = \lambda^2 \Rightarrow \lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda(\lambda-2)=0$

$\therefore \lambda = 2 \; (\because \lambda > 0)$

$\therefore P(X=5)=\dfrac{e^{-2}2^5}{5!}$

3. **In a book of 450 pages, there are 400 typographical errors. Assuming that the number of errors per page follow the Poisson law, find the probability that a random sample of 5 pages will contain no typographical error ?**

Sol:- The average number of errors per page in the book is $\lambda = \dfrac{400}{450}=\dfrac{8}{9}$

The probability that there are 'x' errors per page $\Rightarrow \mathrm{P}(\mathrm{X}=x)=\dfrac{e^{-\lambda}\lambda^x}{x!}=\dfrac{e^{\frac{-8}{9}}\left(\frac{8}{9}\right)^x}{x!}$

$\therefore$ To get no errors, put x = 0 $\Rightarrow P(X=0)=e^{\frac{-8}{9}}$

$\therefore$ Required probability that a random sample of 5 pages will contain no errors is $\left[P(X=0)\right]^5=\left(e^{\frac{-8}{9}}\right)^5$

4. **Find the minimum number of times a fair coin must be tossed so that the probability of getting atleast one head is atleast 0.8.**

Sol:- Let 'n' be the number of times a fair coin tossed.

X denotes the number of heads getting.

X follows binomial distribution with parameters n and $p=\dfrac{1}{2}$

$P(X \geq 1) \geq 0.8$ (Given)

$\Rightarrow 1-P(X=0) \geq 0.8 \;\; \left(\because \; \Sigma P(X=x_i)=1\right)$

$\Rightarrow -P(X=0) \geq 0.8-1$

$\Rightarrow -P(X=0) \geq -0.2$

$\Rightarrow P(X=0) \leq 0.2$

$\Rightarrow {}^nC_0\left(\dfrac{1}{2}\right)^n \leq 0.2$

$\left(\dfrac{1}{2}\right)^n \leq \dfrac{2}{10}$

$\dfrac{1}{2^n} \leq \dfrac{1}{5}$

$\Rightarrow 2^n \geq 5$

It is true for $n \geq 3$

$\therefore$ Minimum value of 'n' is 3.

5. 8 Coins are tossed simultaneously. Find the probability of getting atleast 6 heads.

Sol:- The probability of getting a head $= \frac{1}{2}$

The probability of getting a tail $= \frac{1}{2}$

Probability of getting 'r' heads in a random throw of '8' coins,

$$P(X = x) = {}^8C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{8-x} = {}^8C_x\left(\frac{1}{2}\right)^8, \ x = 0,1,2................8$$

Probability of getting 6 heads is

$$P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8)$$

$$= {}^8C_6\left(\frac{1}{2}\right)^8 + {}^8C_7\left(\frac{1}{2}\right)^8 + {}^8C_8\left(\frac{1}{2}\right)^8$$

$$= \left(\frac{1}{2}\right)^8\left[{}^8C_6 + {}^8C_7 + {}^8C_8\right]$$

$$= \frac{1}{16 \times 16}\left[{}^8C_2 + {}^8C_1 + 1\right]$$

$$= \frac{1}{256}\left[\frac{8 \times 7}{2} + 8 + 1\right] = \frac{1}{256} \times 37 = \frac{37}{256}$$

6. One in 9 ships is likely to be wrecked, when they are set on sail, when 6 ships are on sail, find the probability for

(a) At least one will arrive safely.

(b) Exactly, 3 will arrive safely.

Sol:- Given, one in 9 ships is to be wrecked

Probability of ship to be wrecked $(p) = \frac{1}{9}$

But p+q=1, Put $p = \frac{1}{9}$

$$\frac{1}{9} + q = 1$$

$$q = 1 - \frac{1}{9} = \frac{8}{9}$$

Number of ships are (n)=6

$$P(X = 0) = {}^6C_0\left(\frac{1}{9}\right)^{6-0}\left(\frac{8}{9}\right)^0 = \left(\frac{1}{9}\right)^6$$

a) Probability of atleast one will arive safely

$$= P(X > 0) = 1 - P(X = 0)$$

$$= 1 - \left(\frac{1}{9}\right)^6 = 1 - \frac{1}{9^6}$$

b) Exactly, 3 will arrive safely

$$\Rightarrow P(X=3) = {}^6C_3\left(\frac{1}{9}\right)^{6-3}\left(\frac{8}{9}\right)^3$$

$$=\left(\frac{6\times5\times4}{3\times2}\right)\frac{1}{9^3}\times\frac{8^3}{9^3} = 20\left(\frac{8^3}{9^6}\right)$$

LONG ANSWER TYPE QUESTIONS (7 MARKS)

1. A cubical die is thrown . Find the mean and variance of X, giving the number on the face that shows up.

Sol.: Let S be tha sample space and X be the random variable associated with S, where P(X) is given by the following table.

$X=x_i$	1	2	3	4	5	6
$P(X=x_i)$	1/6	1/6	1/6	1/6	1/6	1/6

Mean of X = $\mu = \sum_{i=1}^{6} x_i P(X=x_i) = 1.\frac{1}{6}+2.\frac{1}{6}+3.\frac{1}{6}+4.\frac{1}{6}+5.\frac{1}{6}+6.\frac{1}{6} = \frac{1}{6}\left(\frac{6\times7}{2}\right)=\frac{7}{2}$

Variance of $X = \sigma^2 = \sum_{i=1}^{6} x_i^2 P(X=x_i) - \mu^2$

$$=1^2.\frac{1}{6}+2^2.\frac{1}{6}+3^2.\frac{1}{6}+4^2.\frac{1}{6}+5^2.\frac{1}{6}+6^2.\frac{1}{6}=\frac{1}{6}\left(\frac{6\times7\times13}{6}\right)-\frac{49}{4}=\frac{35}{12}$$

2. The probability distribution of a random variable of X is given below.

$X=x_i$	1	2	3	4	5
$P(X=x_i)$	k	2k	3k	4k	5k

Find the values of k, mean and variance of X.

Sol:- We have $\sum_{i=1}^{5} P(X=x_i)=1 \Rightarrow k+2k+3k+4k+5k=1 \Rightarrow 15k=1 \Rightarrow k=1/15$

Mean = $\mu = \sum_{i=1}^{5} x_i P(X=x_i)$

$$=1.(k)+2.(2k)+3.(3k)+4.(4k)+5.(5k)=55k=55\times\frac{1}{15}=\frac{11}{3}$$

Variance of $X=(\sigma^2)=\sum_{i=1}^{6} x_i^2 P(X=x_i)-\mu^2$

$$=1^2.(k)+2^2.(2k)+3^2.(3k)+4^2.(4k)+5^2.(5k)-\mu^2$$

$$=k+8k+27k+64k+125k-\left(\frac{11}{3}\right)^2=225k-\frac{121}{9}=225\times\frac{1}{15}-\frac{121}{9}$$

$$=\frac{135-121}{9}=\frac{14}{9}.$$

3. Find the constant 'c' so that $F(x) = c\left(\frac{2}{3}\right)^x$, $x = 1, 2, 3............$ is the p.d.f. ofa discrete random variable 'x' ?

Sol:- Given $F(x) = c\left(\frac{2}{3}\right)^x$

Put x = 1, $F(1) = c\left(\frac{2}{3}\right)^1$

Put x = 2, $F(2) = c\left(\frac{2}{3}\right)^2$

Put x = 3, $F(3) = c\left(\frac{2}{3}\right)^3$

$\sum F(x) = c\sum\left(\frac{2}{3}\right)^x = 1$

$c\left[\left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + ...+ \propto\right] = 1$

It is an infinite G.P. with $r = \frac{2}{3} < 1$

$S_\propto = \frac{a}{1-r}$ where $a = \frac{2c}{3}, r = \frac{2}{3}$

$\Rightarrow c\left[\frac{2/3}{1-2/3}\right] = 1$

$\Rightarrow c\left[{}^{2/3}\!\big/_{1/3}\right] = 1 \quad \Rightarrow c = \frac{1}{2}$

4. Let X be a random variable such that $P(X = -2) = P(X = -1) = P(X = 2) = P(X = 1) = \frac{1}{6}$ and $P(X = 0) = \frac{1}{3}$, Find the mean and variance of X.

Sol:- Mean $(\mu) = \sum_{k=-2}^{2} x_i\, P(X = k)$

$= (-2)\left(\frac{1}{6}\right) + (-1)\frac{1}{6} + (0)\left(\frac{1}{3}\right) + 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right)$

Mean = 0

Variance $\left(\sigma^2\right) = \sum_{k=-2}^{2} (k-\mu)^2 P(X = k)$

$$=\left(-2-0\right)^2\left(\frac{1}{6}\right)+(-1-0)^2\left(\frac{1}{6}\right)+(0-0)^2\left(\frac{1}{3}\right)+(1-0)^2\left(\frac{1}{6}\right)+(2-0)^2\left(\frac{1}{6}\right)$$

$$=(4)\left(\frac{1}{6}\right)+(1)\left(\frac{1}{6}\right)+0+(1)\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)=\frac{4+1+1+4}{6}=\frac{10}{6}=\frac{5}{3}$$

5. A random variable X has the following probability distribution.

$X=x_i$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$P(X=x_i)$	**0**	**k**	**2k**	**2k**	**3k**	k^2	**$2k^2$**	**$7k^2+k$**

Find (i)k (ii) the mean and (iii) P(0<X<5)

Sol:- Sum of all probabilities $\sum P\left(X=x_i\right)=1$

$$\Rightarrow P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7)=1$$

$$\Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$$\Rightarrow 10k^2+9k-1=0$$

$$\Rightarrow 10k^2+10k-k-1=0$$

$$\Rightarrow 10k\left(k+1\right)-1\left(k+1\right)=0$$

$$\Rightarrow \left(10k-1\right)\left(k+1\right)=0$$

$$\Rightarrow 10k-1=0 \quad k\neq -1 \quad \left(\because k>0\right)$$

$$\Rightarrow k=\frac{1}{10}$$

(ii) Mean $(\mu)=\sum_{i=1}^{n} x_i\, P\left(X=x_i\right)$

$$\therefore Mean\ (\mu)=0\left(0\right)+1(k)+2(2k)+3(2k)+4(3k)+5(k^2)+6(2k^2)+7(7k^2+k)$$

$$=0+k+4k+6k+12k+5k^2+12k^2+49k^2+7k$$

$$=66k^2+30k;\ Put\ k=\frac{1}{10}$$

$$=66\left(\frac{1}{10}\right)^2+30\left(\frac{1}{10}\right)=\frac{66}{100}+\frac{3}{1}=\frac{366}{100}$$

$$\mu=3.66$$

(iii) P(0<X<5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)

= k + 2k + 2k + 3k

= 8k

$$= 8\left(\frac{1}{10}\right)$$

$$= \frac{4}{5}$$

6. The range of a random variable X = {0, 1, 2} Given that $P(X=0)=3c^3$, $P(X=1)=4c-10c^2$, $P(X=2)=5c-1$ then

(i) Find the value of 'c' (ii) P(X<1)

(iii) $P(1<X\leq 2)$ and (iv) $P(0<X\leq 3)$

Sol:- Given X = {0, 1, 2}

We know that $\sum_{i=1}^{n} P(X = x_i) = 1$

$\Rightarrow P(X = 0) + P(X = 1) + P(X = 2) = 1$

$\Rightarrow (3c^3) + (4c - 10c^2) + (5c - 1) = 1$

$\Rightarrow 3c^3 - 10c^2 + 9c - 1 - 1 = 0$

$\Rightarrow 3c^3 - 10c^2 + 9c - 2 = 0$

c = 1				
	3	-10	9	-2
	0	3	-7	2
	3	-7	2	0

$\Rightarrow 3c^2 - 7c + 2 = 0$

$\Rightarrow 3c^2 - 6c - c + 2 = 0$

$\Rightarrow 3c(c - 2) - 1(c - 2) = 0$

$\Rightarrow (3c - 1)(c - 2) = 0$

$c = \frac{1}{3}$ $c = 2$

$\therefore c = \frac{1}{3}$

(ii) P(X<1) = P(X=0)

$= 3c^3$

If C=1 $\Rightarrow$ $3c^3=3(1)^3=3>1$ (not possible)

If C=2 $\Rightarrow$ $3c^3=3(2)^3=24>1$ (not possible)

If $C = \frac{1}{3} \Rightarrow 3C^3 = 3\left(\frac{1}{3}\right)^3 = 3\frac{1}{27} = \frac{1}{9} < 1$

$\Rightarrow C = \frac{1}{3}$ and $\therefore P(X < 1) = \frac{1}{9}$

(iii) $P(1<X\leq 2) = P(X=2)$

$= 5C-1$

Put $C=\frac{1}{3}\Rightarrow \quad =5\left(\frac{1}{3}\right)-1=\frac{5}{3}-1=\frac{2}{3}$

(iv) $P(0<x\leq 3) = P(X=1)+P(X=2)+P(X=3)$

$= (4C-10C^2)+(5C-1)+0$

$= -10C^2+9C-1$

Put $C=\frac{1}{3}\Rightarrow \quad =-10\left(\frac{1}{3}\right)^2+9\left(\frac{1}{3}\right)-1$

$=\frac{-10}{9}+3-1$

$=\frac{-10}{9}+\frac{2}{1}=\frac{-10+18}{9}=\frac{8}{9}$

8. The range of a random variable X is {1, 2, 3....} and $P(X=k)=\frac{C^k}{k!};\ (k=1,2,3..........)$

Find the value of (i) 'c' and (ii) $P(0<X<3)$

Sol:- (i) Sum of the probabilities = 1

i.e., $\sum_{i=1}^{\infty} P(X=x_i)=1$

$\Rightarrow P(X=1)+P(X=2)+P(X=3)+..........+\infty=1$

$\Rightarrow \frac{C^1}{1!}+\frac{C^2}{2!}+\frac{C^3}{3!}+..............+\infty=1$

Adding '1' on both sides

$1+\frac{C}{1}+\frac{C^2}{2!}+\frac{C^3}{3!}+.......+\infty=2$

$\Rightarrow e^C=2$

$\Rightarrow C=\log_e 2$ ....(1)

(ii) $P(0<X<3) = P(X=1)+P(X=2)$

$=C+\frac{C^2}{2}$

$=\log_e 2+\frac{(\log_e 2)^2}{2}$ [$\because$ from (1)]

9. Two dice are rolled at random. Find the probability distribution of the sum of the numbers on them. Find the mean of the random variable.

Sol:- Dice has 6 faces and 1 to 6 numbers are written on each face of the dice.

If two dice are rolled, sample space S consists (Total number of outcomes)=6^2=36 points

They are S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1)..........(2, 6),......(6, 6)}

Let X denote the sum of numbers on the two dice

i.e., 1+1=2, 1+2=3, ...6+6=12

$\therefore$ Range of X is X = {2, 3, 4, 5...........12}

Probability distribution for X is given below.

$X = X_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X=X_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean of $X = \mu = \sum_{i=2}^{12} x_i P(X = x_i)$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right)$$

$$+7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$$

$$= \frac{1}{36}[2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12]$$

$$= \frac{252}{36} = 7$$

10.

X=x	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
P(X=x)	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

is the probability distribution of a random variable 'X'. Find the variance of X.

Sol:- Mean $(\mu) = \sum_{i=-3}^{3} x_i P(X = x_i)$

$$= (-3)\left(\frac{1}{9}\right) + (-2)\left(\frac{1}{9}\right) + (-1)\left(\frac{1}{9}\right) + (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{9}\right) + (2)\left(\frac{1}{9}\right) + (3)\left(\frac{1}{9}\right)$$

$$= \frac{1}{9}[-3 - 2 - 1 + 0 + 1 + 2 + 3]$$

$$\mu = \frac{1}{9}[0] = 0$$

Variance $\left(\sigma^2\right)=\sum_{i=-3}^{3} x_i^{\,2} P\left(X=x_i\right)-\mu^2$

$$=(-3)^2\left(\frac{1}{9}\right)+(-2)^2\left(\frac{1}{9}\right)+(-1)^2\left(\frac{1}{9}\right)+(0)^2\left(\frac{1}{3}\right)+(1)^2\left(\frac{1}{9}\right)+(2)^2\left(\frac{1}{9}\right)+(3)^2\left(\frac{1}{9}\right)-0$$

$$=(9)\left(\frac{1}{9}\right)+4\left(\frac{1}{9}\right)+\left(\frac{1}{9}\right)+0+\left(\frac{1}{9}\right)+4\left(\frac{1}{9}\right)+9\left(\frac{1}{9}\right)$$

$$=1+\frac{4}{9}+\frac{1}{9}+\frac{1}{9}+\frac{4}{9}+1$$

$$\sigma^2=2+\frac{4+1+1+4}{9}=2+\frac{10}{9}=\frac{28}{9}$$

11.

X=x	**-2**	**-1**	**0**	**1**	**2**	**3**
P(X=x)	**0.1**	**k**	**0.2**	**2k**	**0.3**	**k**

is the probability distribution of a random variable X. Find the value of 'k' and the variance of X ?

Sol:- Sum of all probabilities = 1

0.1+k+0.2+2k+0.3+k=1

4k+0.6=1

4k=1-0.6

4k=0.4

$$k=\frac{0.4}{4}=0.1$$

Mean $(\mu)=\sum x_i.P\left(X=x_i\right)$

$$=(-2)(0.1)+(-1)(k)+(0)(0.2)+(1)(2k)+2(0.3)+3(k)$$

$$=-0.2-k+0+2k+0.6+3k$$

$$=4k+0.4$$

$$=4(0.1)+0.4$$

$$=0.4+0.4$$

$$\mu=0.8$$

Variance $\left(\sigma^2\right)=\sum x_i^2.P\left(X=x_i\right)-\mu^2$

$$=(-2)^2(0.1)+(-1)^2(k)+(0)^2(0.2)+(1)^2(2k)+(2)^2(0.3)+(3)^2(k)-(0.8)^2$$

$= 4(0.1) + k + 0 + 1(2k) + 4(0.3) + 9k - 0.64$

$= 0.4 + k + 2k + 1.2 + 9k - 0.64$

$= 12k + 0.96$

$= 12(0.1) + 0.96$

$= 1.2 + 0.96$

Variance $(\sigma^2) = 2.16$

Problems for Practice

(i) If X is a random variable with probability distribution

$P(X = k) = \frac{(k+1)c}{2^k}, \; k = 0,1,2........$ then find 'c' ?

Ans:- $C = \frac{1}{4}$ (Hint : Refer Example 3 from text book page No. 353)

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