

## Locus

**Def :** The path of a point moving subject to the given condition is called locus.

The equation representing the locus is called equation of locus.

### Steps to find locus

**Step 1 :** Let the point P be  $P(h, k)$

**Step 2 :** Use the formula related to given condition

**Step 3 :** Simplify

**Step 4 :** The simplified algebraic equation represents locus

### Short Answer Questions (4 Marks)

1. Find the equation of locus of a point which is at a distance 5 from  $A(4, -3)$ .

**Sol:** Let  $P(h, k)$  be any point

Given point  $A(4, -3)$

$|PA| = 5$  given condition

$$|PA|^2 = 25$$

$$(h - 4)^2 + (k + 3)^2 = 25 \quad \text{distance between two points} \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{equation of locus } (h - 4)^2 + (k + 3)^2 = 25$$

$$h^2 + k^2 - 8h - 6k + 25 = 25 \quad x^2 + y^2 - 8x - 6y = 0$$

2. Find the equation of locus of a point which is equidistant from the points  $A(-3, 2)$ , and  $B(0, 4)$

**Sol:** Let  $P(h, k)$  be any point

$|PA| = |PB|$  given condition

$$|PA|^2 = |PB|^2$$

$$(h + 3)^2 + (k - 2)^2 = h^2 + (k - 4)^2$$

$$h^2 + 6h + 9 + k^2 - 4k + 4 = h^2 + k^2 - 8k + 16$$

$$6h^2 + 4k^2 - 13x + 8y - 16 = 0$$

$$6h^2 + 4k^2 - 3 = 0$$

$$\text{equation of locus } 6x^2 + 4y^2 - 3 = 0$$

3. Find the equation of locus of a point P such that the distance of P from the origin is twice the distance of P and A(1,2).

Sol: Let P(h, k) be any point

O(0, 0) be the origin

A(1, 2) given point

$|PO| = 2|PA|$  given condition

$$|PO|^2 = 4|PA|^2$$

$$h^2 + k^2 = 4[(h-1)^2 + (k-2)^2]$$

$$\text{simplifying } 3h^2 + 3k^2 - 8h - 16k + 20 = 0$$

$$\text{equation of locus } 3x^2 + 3y^2 - 8x - 16y + 20 = 0$$

4. Find the equation of locus of a point which is equidistant from the coordinate axes.

Sol: Let P(h, k) be any point

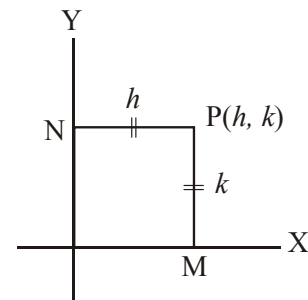
distance from X-axis =  $|PM|$

distance from Y-axis =  $|PN|$

$$|PM| = |PN| \quad |PM|^2 = |PN|^2$$

$$k^2 = h^2.$$

$$\text{equation of locus } y^2 = x^2 \quad x^2 = y^2$$



5. Find the equation of locus of a point equidistant from A(2,0) and the Y-axis.

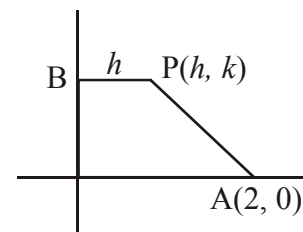
Sol: Let P(h, k) be any point

given point = A(2, 0)

given condition  $|PA| = h$

$$|PA|^2 = h^2 \quad (h-2)^2 + k^2 = h^2$$

$$\text{equation of locus } y^2 - 4x + 4 = 0.$$



6. Find the equation of locus of a point P such that  $|PA|^2 + |PB|^2 = 2c^2$  where A(a, 0), B(-a, 0) and  $0 < |a| < |c|$

Sol: Let P(h, k) be any point

given point = A(a, 0), B(-a, 0)

given condition  $|PA|^2 + |PB|^2 = 2c^2$

$$(h-a)^2 + k^2 + (h+a)^2 + k^2 = 2c^2$$

$$\text{simplifying } h^2 + k^2 = c^2 - a^2$$

$$\text{Equation of locus } x^2 + y^2 = c^2 - a^2.$$

7. Find the equation of locus of P, if the line segment joining (2, 3) and (-1, 5) subtends a right angle at P.

Sol: Let P(h, k) be any point

given point = A(2, 3), B(-1, 5)

$$\text{slope of PA } m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{k - 3}{h - 2}$$

$$\text{slope of PB } m_2 = \frac{k - 5}{h - 1}$$

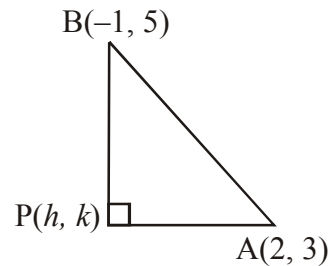
PA ⊥ PB

$$m_1 \cdot m_2 = -1$$

$$\frac{k - 3}{h - 2} \cdot \frac{k - 5}{h - 1} = -1$$

simplifying  $h^2 + k^2 - h - 8k + 13 = 0$

equation of locus  $x^2 + y^2 - x - 8y + 13 = 0$ .



8. The ends of the hypotenuse of a right angle triangle are (0,6) and (6,0). Find the equation of the locus of its third vertex.

Sol: Let P(h, k) be any point

given point = (6, 0), (0, 6)

$$\text{slope of PA } m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{k - 0}{h - 0} = \frac{k}{h}$$

$$\text{slope of PB } m_2 = \frac{k - 6}{h - 0} = \frac{k - 6}{h}$$

given condition PA ⊥ PB

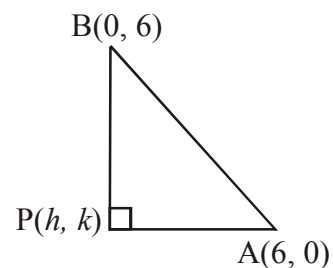
$$m_1 \cdot m_2 = -1$$

$$\frac{k}{h} \cdot \frac{k - 6}{h} = -1$$

$$k^2 - 6k = -[h^2 - 6h]$$

simplifying  $h^2 + k^2 - 6h - 6k = 0$ .

equation of locus  $x^2 + y^2 - 6x - 6y = 0$ .



9. Find the equation of the locus of a point, the difference of whose distance from (-5, 0) and (3, 0) is 8.

Sol: Let P(h, k) be any point

given point = A(-5, 0), B(3, 0)

given condition  $|PA - PB| = 8$

$$|PA| = 8 + |PB|$$

squaring both sides

$$|PA|^2 = [8 + |PB|]^2$$

simplifying  $|PA|^2 - |PB|^2 - 64 = 16 |PB|$

$$[(h - 5)^2 + k^2] - [(h - 3)^2 + k^2] - 64 = 16 |PB|$$

$$20h - 64 = 16 |PB|$$

$$5h - 16 = 4 |PB|$$

$$(5h - 16)^2 = 16 |PB|^2$$

$$(5h - 16)^2 = 16 [(h - 3)^2 + k^2]$$

simplifying  $\frac{h^2}{16} - \frac{k^2}{9} = 1$

$$\text{equation of locus } \frac{x^2}{16} - \frac{y^2}{9} = 1.$$

**10. Find the equation of the locus of P, if A(4, 0), B(-4, 0) and  $|PA - PB| = 4$**

**Sol:** Let P(h, k) be any point

given points = A(4, 0), B(-4, 0)

given condition  $|PA - PB| = 4$

$$|PA| = 4 + |PB|$$

$$[|PA|^2 - |PB|^2] - 16 = 8 |PB|$$

$$[(h - 4)^2 + k^2] - [(h + 4)^2 + k^2] - 16 = 8 |PB|$$

$$-16h - 16 = 8 |PB|$$

$$4(h + 1) = (h + 4) + k$$

simplifying  $3h^2 - k^2 = 12$

$$\text{equation of locus } \frac{x^2}{4} - \frac{y^2}{12} = 1.$$

**11. Find the equation of the locus of a point, the sum of whose distances from (0,2) and (0,-2) is 6.**

**Sol:** Let P(h, k) be any point

given points = A(2, 3), B(2, -3)

given condition  $|PA| + |PB| = 8$

$$|PA| = 8 - |PB|$$

$$[|PA|^2 - |PB|^2] - 64 = -16 |PB|$$

$$[(h - 2)^2 + (k - 3)^2] - [(h - 2)^2 + (k + 3)^2] - 64 = -16 |PB|$$

$$-12k - 64 = -16 |PB|$$

$$(3k + 16 = 16[(h - 2) + (k + 3)])$$

simplifying  $16h^2 + 7k^2 - 64h - 48 = 0$

equation of locus  $16x^2 + 7y^2 - 64x - 48 = 0.$

12. **A(5, 3), B(3, -2) are two fixed points. Find the equation of the locus of P, so that the area of triangle PAB is 9.**

**Sol:** Let 3rd vertex be P(h, k)

given points = A(5, 3), B(3, -2)

given condition  $|PA| + |PB| = 8$

$$\text{area of } \triangle PAB = \frac{1}{2} \begin{vmatrix} h & k \\ 5 & 3 \\ 3 & 2 \\ h & k \end{vmatrix} = 9$$

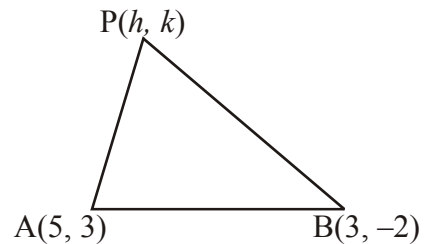
$$|3h \ 5k \ 10 \ 9 \ 3k \ 2h| = 18$$

$$|5h \ 2k \ 19| = 18$$

$$(5h \ 2k \ 1)(5h \ 2k \ 37) = 0$$

equation of locus  $(5h \ 2k \ 1)(5h \ 2k \ 37) = 0.$

$$(5x \ 2y \ 1)(5x \ 2y \ 37) = 0$$



**Practice Problem**

**A(2,3) and B(-3,4) are two given points. Find the equation of locus of P so that the area of the triangle PAB is 8.5**

13. **If the distance from P to the points (2, 3), (2, -3) are in the ratio 2 : 3, then find the equation of the locus of P.**

**Sol:** Let (h, k) are the vertices of the triangle

Let P(h, k) be any point

Given points = A(2, 3), B(2, -3)

given condition  $\frac{|PA|}{|PB|} = \frac{2}{3}$

$$3|PA| = 2|PB|$$

$$9|PA|^2 = 4|PB|^2$$

$$9[(h - 2)^2 + (k - 3)^2] = 4[(h - 2)^2 + (k + 3)^2]$$

simplifying  $5h^2 + 5k^2 - 20h - 78k + 65 = 0$

equation of locus  $5x^2 + 5y^2 - 20x - 78y + 65 = 0.$

14. A(1, 2), B(2, -3) and C(-2, 3) are three points. A point P moves such that  $|PA|^2 + |PB|^2 = 2|PC|^2$ . Show that the equation to the locus of P is  $7x - 7y + 4 = 0$ .

**Sol:** Let P(h, k) be any point

$$|PA|^2 + |PB|^2 = (h - 1)^2 + (k - 2)^2 + (h - 2)^2 + (k + 3)^2$$

$$2|PC|^2 = 2[(h + 2)^2 + (k - 3)^2]$$

$$\text{substituting in } |PA|^2 + |PB|^2 = 2|PC|^2$$

$$\text{simplifying } 7h - 7k + 4 = 0$$

$$\text{equation of locus } 7x - 7y + 4 = 0.$$

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## Transformation of Axis

**Def :** Without changing the direction of co-ordinate axes of the origin is shifted to a given point then the change occurred is called translation of axes.

Let  $P(x, y)$  original coordinates

and  $P(X, Y)$  transformed coordinates

$$x = PQ = ON = OL + LN = OL + O'M = h + X = X + h$$

$$y = PN = PM + MN = Y + O'L = Y + k$$

$$x = X + h, \quad y = Y + k \quad X = x - h, \quad Y = y - k$$

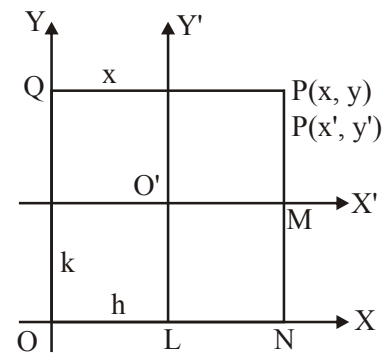
$$P(x, y) = (X + h, Y + k)$$

$$P(X, Y) = (x - h, y - k)$$

$$(h, k) = (x - X, y - Y)$$

original equation of the curve  $f(x, y)$

transformed equation of the curve  $f(X, Y)$



translation of axis

**Rotation of axes :** without the changing the position of origin the axes are rotated through an angle then it is called rotation of axes.

$$x = OL = OQ - LQ = X \cos \theta - NM$$

$$= X \cos \theta - Y \sin \theta$$

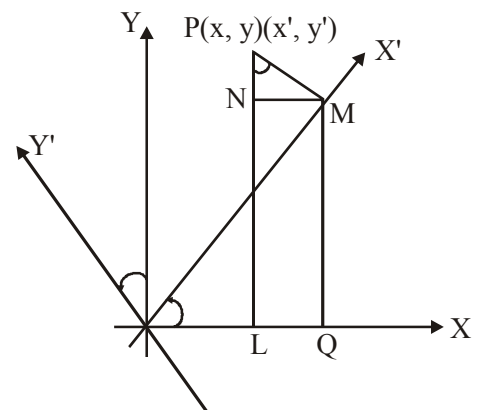
$$y = PL = PN + NL = PN + MQ$$

$$= PM \cos \theta + OM \sin \theta$$

$$= Y \cos \theta + X \sin \theta = X \sin \theta + Y \cos \theta$$

$$P(x, y) = (X \cos \theta - Y \sin \theta, X \sin \theta + Y \cos \theta)$$

	X	Y
x	$\cos \theta$	$-\sin \theta$
y	$\sin \theta$	$\cos \theta$



Rotation of axis

### Short Answer Questions (4 Marks)

1. When the origin is shifted to  $(-1, 2)$  by the translation of axes, find the transformed equations of the following.

**Sol :** (1)  $x^2 + y^2 - 2x - 4y - 1 = 0$

$(h, k) = (-1, 2)$

$x = X + h = X - 1$

$y = Y + k = Y + 2$

substituting  $(X - 1)^2 + (Y + 2)^2 - 2(X - 1) - 4(Y + 2) - 1 = 0$

simplifying  $X^2 + Y^2 - 4 = 0$  required equation

(2)  $2x^2 + y^2 - 4x - 4y = 0$

$x = X + h = X - 1$

$y = Y + k = Y + 2$

substituting  $2(X - 1)^2 + (Y + 2)^2 - 4(X - 1) - 4(Y + 2) = 0$

simplifying  $2[X^2 - 2X + 1] + [Y^2 + 4Y + 4] - 4X + 4 - 4Y - 8 = 0$

$2X^2 + Y^2 - 8X - 8Y - 18 = 0$  required equation

2. The point to which the origin is shifted and the transformed equation are given below. Find the original equation.

**Sol :** (1)  $(3, -4), x^2 + y^2 = 4$

Let the transformed equation indicated by  $X^2 + Y^2 = 4$

simplifying  $(h, k) = (3, -4)$

$X = x - h \quad Y = y - k$

$X = x - 3 \quad Y = y + 4$

original equation after substitution and simplification

$x^2 + y^2 - 6x - 8y - 21 = 0$

(2)  $(-1, 2), x^2 + y^2 - 16 = 0$

changed origin is  $(h, k) = (-1, 2)$

given transformed equation  $x^2 + 2y^2 - 16 = 0$

Let this denoted by  $X^2 + 2Y^2 - 16 = 0$

$X = x - h = x + 1 \quad Y = y - k = y - 2$

original equation after substitution and simplification

$x^2 + 2y^2 - 2x - 8y - 25 = 0$

3. Find the point to which the origin is to be shifted so as to remove the first degree terms from the equation.

**Sol :** given equation  $f(x, y) = 4x^2 + 9y^2 - 8x - 36y - 4 = 0$



transformed equation

$$f(X-h, Y-k) = f(X, Y) = 4(X-h)^2 - 9(Y-k)^2 - 8(X-h) + 36(Y-k) - 4 = 0$$

To eliminate X, Y terms equate coefficient of X and coefficient of Y to 0

$$8h - 8 = 0 \quad h = 1$$

$$18k + 36 = 0 \quad k = -2$$

the point to which the origin to be shifted is  $(h, k) = (1, -2)$  or

use the formula  $\frac{hf}{ab} - \frac{bg}{h^2}, \frac{gh}{ab} - \frac{af}{h^2}$

4. When the origin is shifted to  $(2, 3)$  then the transformed equation of a curve is  $x^2 + 3xy - 2y^2 - 17x - 7y - 11 = 0$ , then find the original equation of the curve.

Sol : transformed equation  $x^2 - 3xy - 2y^2 - 17x - 7y - 11 = 0$

for convenient let the equation  $X^2 + 3XY - 2Y^2 + 17X - 7Y - 11 = 0$

$$(h, k) = (2, 3)$$

$$X = x - h = x - 2 \quad Y = y - k = y - 3$$

substituting in (1)

$$(x-2)^2 - 3(x-2)(y-3) - 2(y-3)^2 - 17(x-2) - 7(y-3) - 11 = 0$$

original equation  $x^2 - 3xy - 2y^2 - 4x - y - 20 = 0$

1. When the axes are rotated through an angle  $\frac{\pi}{6}$ , find the transformed equation of

$$x^2 + 2\sqrt{3}xy - y^2 = 2a^2$$

Sol :  $\theta = \frac{\pi}{6}$

$$f(x, y) = x^2 + 2\sqrt{3}xy - y^2 - 2a^2$$

$$x = X \cos \frac{\pi}{6} - Y \sin \frac{\pi}{6} = X \cos \frac{\pi}{6} - Y \sin \frac{\pi}{6} = \frac{X\sqrt{3}}{2} - \frac{Y}{2}$$

$$y = X \sin \frac{\pi}{6} + Y \cos \frac{\pi}{6} = X \sin \frac{\pi}{6} + Y \cos \frac{\pi}{6} = \frac{X}{2} + \frac{Y\sqrt{3}}{2}$$

transformed equation =  $f(X, Y)$

$$\left(\frac{\sqrt{3}X - Y}{2}\right)^2 + 2\sqrt{3} \left(\frac{\sqrt{3}X - Y}{2}\right) \left(\frac{X + Y\sqrt{3}}{2}\right) - \left(\frac{X + Y\sqrt{3}}{2}\right)^2 = 2a^2$$

simplifying  $X^2 - Y^2 = a^2$ .

2. When the axes are rotated through an angle  $\pi/4$ , find the transformed equation of  $3x^2 - 10xy + 3y^2 = 9$

Sol :  $\theta = \pi/4$

$$f(x, y) = 3x^2 - 10xy + 3y^2 = 9$$

$$x = X \cos \theta - Y \sin \theta = \frac{X - Y}{\sqrt{2}}$$

$$y = X \sin \theta + Y \cos \theta = \frac{X + Y}{\sqrt{2}}$$

by substituting and simplifying the transformed equation  $8X^2 - 2Y^2 = 9$

3. When the axes are rotated through an angle  $\alpha$ , find the transformed equation of  $x \cos \alpha + y \sin \alpha = p$

Sol :  $x = X \cos \theta - Y \sin \theta$

$$y = X \sin \theta + Y \cos \theta$$

by substituting

$$(X \cos \theta - Y \sin \theta) \cos \theta + (X \sin \theta + Y \cos \theta) \sin \theta = p$$

$$X(\cos^2 \theta + \sin^2 \theta) = p$$

$$X = p.$$

4. When the axes are rotated through an angle  $45^\circ$ , the transformed equation of a curve is  $17x^2 - 16xy + 17y^2 = 225$ . Find the original equation of the curve.

Sol : angle of rotation  $\theta = 45^\circ$

let the transformed equation be

$$17X^2 - 16XY + 17Y^2 = 225$$

$$X = x \cos \theta + y \sin \theta \quad Y = -x \sin \theta + y \cos \theta$$

$$\frac{x + y}{\sqrt{2}} \quad \frac{-x + y}{\sqrt{2}}$$

simplifying

$$17 \left( \frac{x + y}{\sqrt{2}} \right)^2 - 16 \left( \frac{x + y}{\sqrt{2}} \right) \left( \frac{-x + y}{\sqrt{2}} \right) + 17 \left( \frac{-x + y}{\sqrt{2}} \right)^2 = 225$$

$$\text{required equation } 25x^2 - 9y^2 = 225$$

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## The Straight Line

### Chapter 3(a)

#### Note :

If a non - vertical straight line makes an angle '  $\theta$  ' with the X- axis measured counter - clock wise from the positive direction of X - axis, then  $\tan \theta$  is called the slope of the line L.

$$m = \tan \theta$$

Slope of X- axis & its parallel line is zero

Slope of Y- axis & its parallel line is not define

A line which is passing through  $A(x_1, y_1)$ ,  $B(x_2, y_2)$  then its slope ( $m$ )  $= \frac{y_2 - y_1}{x_2 - x_1}$

The equation of the S.L. which cut off non - zero intercepts 'a' and 'b' on the X-axis and the

Y-axis respectively is  $\frac{x}{a} + \frac{y}{b} = 1$

#### Intercept Form

6. The equation of the S.L. with slope 'm' and cutting off y - intercept 'C' is  $y = mx + c$  slope - intercept form

If it passes through origin then the equation is  $y = mx$

8. **Point - slope form :**

The equation of the straight line with slope 'm' and passing through the point  $(x_1, y_1)$  is

$$y - y_1 = m(x - x_1)$$

The equation of the S.L. passing through the points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \quad \text{Two point form}$$

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) \text{ are collinear} \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad \text{OR}$$

Slope of AC = Slope of AB OR Area of ABC

### Questions

1. Find the equation of the S.L. passing through the points  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$

Sol:  $A(at_1^2, 2at_1), B(at_2^2, 2at_2)$

$$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)}$$

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} (x - at_1^2)$$

$$y - 2at_1 = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} (x - at_1^2)$$

$$y - 2at_1 = \frac{2(t_2 - t_1)}{(t_2 - t_1)(t_2 + t_1)} (x - at_1^2)$$

$$(t_2 - t_1)(y - 2at_1) = 2(x - at_1^2)$$

$$(t_1 - t_2)y - (t_1 - t_2)2at_1 = 2x - 2at_1^2$$

$$2x - 2at_1^2 - (t_1 - t_2)y + (t_1 - t_2)2at_1 = 0$$

$$2x - 2at_1^2 - (t_1 - t_2)y + 2at_1^2 - 2at_1t_2 = 0$$

$$2x - (t_1 - t_2)y - 2at_1t_2 = 0$$

2. Find the value of x, if the slope of the line passing through (2, 5) and (x, 3) is 2.

Sol:  $A(2, 5), B(x, 3), m = 2$

$$\boxed{\text{Slope of AB} = \frac{y_2 - y_1}{x_2 - x_1}}$$

$$\frac{3 - 5}{x - 2} = 2$$

$$-2 = 2(x - 2)$$

$$2x - 4 = -2$$

$$2x = -2 + 4 \quad 2x = 2 \quad x = 1.$$

3. Find the value of  $y$  if the line joining points  $(3, y)$  and  $(2, 7)$  is parallel to the line the points  $(-1, 4)$  and  $(0, 6)$

Sol:  $A(3, y), B(2, 7), C(-1, 4), D(0, 6)$

Slope of AB = Slope of CD

$$\frac{7 - y}{2 - 3} = \frac{6 - 4}{0 - (-1)} \quad \frac{7 - y}{-1} = \frac{2}{1}$$

$$7 - y = -2 \quad 7 + 2 = y \quad y = 9.$$

4. Find the condition for the points  $(a, 0)$  and  $(0, b)$ , where  $ab \neq 0$ , to be collinear

Sol:  $A(a, 0), B(h, k), C(0, b)$

points A, B & C are collinear      Slope of AC = Slope of AB

$$\frac{b - 0}{0 - a} = \frac{k - 0}{h - a} \quad \frac{b}{-a} = \frac{k}{h - a}$$

$$b(h - a) = ak$$

$$bh - ab = ak \quad ak - bh = ab$$

$$\frac{ak}{ab} - \frac{bh}{ab} = \frac{ab}{ab} \quad \frac{h}{a} - \frac{k}{b} = 1$$

5. Find the equation of the S.L. which makes the  $150^\circ$  with the position X-axis in the positive direction and which passes through the point  $(-2, -1)$

Sol:  $\theta = 150^\circ, A(-2, -1)$

$$m = \tan 150^\circ = \tan(90 + 60) = -\cot 60^\circ = -\frac{1}{\sqrt{3}}$$

The equation of the S.L. with slope  $-\frac{1}{\sqrt{3}}$  and passing through the point  $(-2, -1)$  is

$$y - y_1 = m(x - x_1)$$

$$y + 1 = -\frac{1}{\sqrt{3}}(x + 2)$$

$$\sqrt{3}(y + 1) = -(x + 2)$$

$$\sqrt{3}y + \sqrt{3} = -(x + 2)$$

$$\sqrt{3}(y + 1) = -x - 2$$

$$x + \sqrt{3}y + 2 + \sqrt{3} = 0$$

**Exercise:**

Find the slopes of the lines (i) parallel to and (ii) perpendicular to the line passing through (6, 3) and (-4, 5)

6. The angle made by a S.L. with the positive X-axis in the positive direction is  $60^\circ$  and the Y- intercept cut off by it is 3. Find the equation of the S.L

Sol:  $\theta = 60^\circ, C = 3$

$$\text{Required equation } y = mx + c \quad m = \tan 60^\circ = \sqrt{3}$$

$$y = \sqrt{3}x + 3$$

$$\sqrt{3}x - y + 3 = 0$$

7. Find the equation of the S.L. passing through the origin and making equal angles with the coordinate axes.

Sol: O(0, 0)

Since making equal angles with the coordinate axes

$$\theta = 45^\circ, 135^\circ.$$

$$m = \tan 45^\circ \text{ or } \tan 135^\circ$$

$$m = 1 \text{ or } \tan 135^\circ = \tan(180^\circ - 45^\circ) = -\tan 45^\circ = -1$$

$$m = \pm 1$$

Equation of S.L. with slope  $\pm 1$  and passing through origin is

$$(y - 0) = \pm 1(x - 0)$$

$$y = \pm x$$

$$x + y = 0 \text{ or } x - y = 0$$

8. Find the equation of the S.L. passing through the point (2, 3) and making non - zero intercepts on the axes of coordinatas whose sum is zero

Sol: A(2, 3)

sum of the intercepts = zero

$$a + b = 0 \quad b = -a$$

Equation of a S.L. in intercept form

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \frac{x}{a} + \frac{y}{-a} = 1$$

$$\frac{x - y}{a} = 1 \quad x - y = a$$

But it is passing through A(2, 3)

$$2 - 3 = a \quad a = -1$$

From

$$x - y = -1$$

$$x - y + 1 = 0$$

9. Find the equation of the S.L. passing through (-4, 5) and cutting off equal and non zero intercepts on the coordinate axes.

Sol: A = (-4, 5)

According to the problem a = b  
equation of a S.L. in intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{a} + \frac{y}{a} = 1$$

$$\frac{x+y}{a} = 1 \quad x+y = a$$

But it is passing through A(-4, 5)

$$-4 + 5 = a \quad a = 1$$

Equation of the required S.L. is

$$x + y = 1$$

$$x + y - 1 = 0$$

10. Find the equation of the S.L. passing through A(-1, 3) and (i) parallel (ii) perpendicular to the S.L. passing through B(2, -5) and C(4, 6)

Sol: A(-1, 3), B(2, -5), C(4, 6)

$$\text{Slope of BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - (-5)}{4 - 2} = \frac{6 + 5}{2} = \frac{11}{2}$$

(i) Slope of a parallel line to BC also  $\frac{11}{2}$

equation of S.L. with slope  $\frac{11}{2}$  and passing through A(-1, 3) is

$$y - 3 = \frac{11}{2}(x - (-1))$$

$$2(y - 3) = 11(x + 1)$$

$$2y - 6 = 11x + 11$$

$$11x - 2y + 17 = 0$$

(ii) Slope of a line which is perpendicular to BC is  $-\frac{1}{m}$

$$-\frac{1}{\frac{11}{2}} = -\frac{2}{11}$$

equation of a S.L. with slope  $\frac{2}{11}$  and passing through A(-1, 3) is

$$y - 3 = \frac{2}{11}(x - (-1))$$

$$11(y - 3) = 2(x + 1)$$

$$11y - 33 = 2x + 2$$

$$2x - 11y + 31 = 0$$

## The Straight Line

### Exercise 3b

#### Note :

The equation of the straight line, whose distance from the origin is P and the normal ray of which drawn from the origin makes an angle  $\alpha$  with the positive direction of the X-axis measured counter clock - wise is

$$x \cos \alpha + y \sin \alpha = P$$

It is called Normal Form

The equation of the straight line passing through  $(x_1, y_1)$  and making an angle  $\alpha$  with the positive direction of the X-axis measured counter - clock wise is

$$(x - x_1) \cos \alpha + (y - y_1) \sin \alpha = 0$$

$$\frac{x - x_1}{\cos \alpha} + \frac{y - y_1}{\sin \alpha} = r \text{ say}$$

$x - x_1 = r \cos \alpha$ ,  $y - y_1 = r \sin \alpha$  is called parameteric form of the point P.

$|r|$  denotes the distance of the point  $(x_1, y_1)$  from the point  $(x, y)$  on the straight line.

General form of a straight line is  $ax + by + c = 0$

$$\text{Its slope } (m) = -\frac{a}{b}$$

## Questions

### 1. Transform the equation $x + y + 1 = 0$ into normal form

**Sol:**  $x + y + 1 = 0$

Normal form of  $ax + by + c = 0$  is

$$\frac{ax}{\sqrt{a^2 + b^2}} + \frac{(b)y}{\sqrt{a^2 + b^2}} + \frac{c}{\sqrt{a^2 + b^2}}, \quad \text{where } c > 0$$

$$\text{Divide with } \sqrt{a^2 + b^2} \quad \sqrt{(1)^2 + (1)^2} \quad \sqrt{1 + 1} \quad \sqrt{2}$$



$$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}}$$

Compare with  $x \cos \alpha + y \sin \alpha = P$

$$\cos \alpha = \frac{1}{\sqrt{2}}, \sin \alpha = \frac{1}{\sqrt{2}}, P = \frac{1}{\sqrt{2}}$$

$$\frac{5}{4}$$

Equation of the required straight line

$$x \cos \frac{5}{4} + y \sin \frac{5}{4} - \frac{1}{\sqrt{2}}$$

2. Transform the equation  $4x - 3y + 12 = 0$  into (a) slope - intercept form (b) intercept form and (c) normal form

Sol. :  $L = 4x - 3y + 12 = 0$

- (a) Slope - intercept form  $y = mx + c = 0$

$$4x - 12 - 3y = 0$$

$$y = \frac{4x - 12}{3}$$

$$y = \frac{4}{3}x - 4$$

$$m = \frac{4}{3}, c = -4$$

- (b) Intercept form  $\frac{x}{a} + \frac{y}{b} = 1$

$$4x - 3y - 12 = 0$$

$$4x - 3y = 12$$

$$\frac{4x}{12} - \frac{3y}{12} = \frac{12}{12}$$

$$\frac{x}{3} - \frac{y}{4} = 1$$

$$a = 3, b = -4$$

- (c) Normal form  $(x \cos \alpha + y \sin \alpha = P)$

$$4x - 3y - 12 = 0$$

$$4x + 3y = 12$$

Divide with  $\sqrt{a^2 + b^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{16 + 9} = 5$

$$\frac{4x}{5} + \frac{3y}{5} = \frac{12}{5}$$

$$\cos = \frac{4}{5}, \sin = \frac{3}{5}, P = \frac{12}{5}$$

3. Transform the equation  $\sqrt{3}x + y = 4$  into (a) slope intercept form (b) intercept form and (c) normal form.

Sol. :  $L = \sqrt{3}x + y = 4$

- (a) Slope - intercept form  $y = mx + c$

$$\sqrt{3}x + y = 4$$

$$y = -\sqrt{3}x + 4$$

$$m = -\sqrt{3}, c = 4$$

- (b) Intercept form  $\frac{x}{a} + \frac{y}{b} = 1$

$$\sqrt{3}x + y = 4$$

$$\frac{\sqrt{3}}{4}x + \frac{y}{4} = \frac{4}{4}$$

$$\frac{x}{\frac{4}{\sqrt{3}}} + \frac{y}{4} = 1$$

$$a = \frac{4}{\sqrt{3}}, b = 4$$

- (c) Normal form  $(x \cos \theta + y \sin \theta = P)$

$$\sqrt{3}x + y = 4$$

Divide with  $\sqrt{a^2 + b^2} = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3 + 1} = 2$

$$\frac{\sqrt{3}x}{2} + \frac{y}{2} = \frac{4}{2}$$

$$x \frac{\sqrt{3}}{2} + y \frac{1}{2} = 2$$

$$\cos = \frac{\sqrt{3}}{2}, \sin = \frac{1}{2}, P = 2$$

$$30^\circ = \frac{\pi}{6}$$

$$x \cos \frac{\pi}{6} = y \sin \frac{\pi}{6} = 2$$

**Home Work**

II 3i, iv, v, vi

4. A straight line parallel to the line  $y = \sqrt{3}x$  passes through  $Q(2, 3)$  and cuts the line  $2x + 4y - 27 = 0$  at P. Find the length PQ

Sol:  $y = \sqrt{3}x$  .....(1)       $2x + 4y - 27 = 0$  .....(2)

Slope of a equation which is parallel to  $y = \sqrt{3}x$  is also  $\sqrt{3}$ .

$$m = \sqrt{3} = \tan 60^\circ$$

$$= 60^\circ \text{ and } Q(2, 3)$$

$$P(x, y) = (x_1 + r \cos \theta, y_1 + r \sin \theta)$$

$$P(2 + r \cos \theta, 3 + r \sin \theta)$$

$$2 + r \frac{1}{2}, 3 + r \frac{\sqrt{3}}{2}$$

$$2 + \frac{r}{2}, 3 + \frac{\sqrt{3}r}{2}$$

But P is on equation

$$2 + \frac{r}{2} + 4 \left( 3 + \frac{\sqrt{3}r}{2} \right) - 27 = 0$$

$$2 + \frac{4r}{2} + 12 + \frac{6\sqrt{3}r}{2} - 27 = 0$$

$$4r + 12 + 2\sqrt{3}r - 27 = 0$$

$$(2\sqrt{3} + 4)r = 15$$

$$r = \frac{15}{2\sqrt{3} + 4} = \frac{15}{2\sqrt{3} + 4} \cdot \frac{2\sqrt{3} - 4}{2\sqrt{3} - 4} = \frac{15(2\sqrt{3} - 4)}{12 - 16}$$

$$r = \frac{15(2\sqrt{3} - 4)}{-4} = \frac{15(4 - 2\sqrt{3})}{4}$$

$$PQ = |r| = \frac{15(4 - 2\sqrt{3})}{4}$$

5. If the area of the triangle formed by the straight lines  $x = 0$ ,  $y = 0$  and  $3x + 4y = a$  ( $a > 0$ ) is 6, find the value of  $a$ .

Sol:  $3x + 4y = a$

$$\frac{3x}{a} + \frac{4y}{a} = \frac{a}{a} \quad \frac{x}{\frac{a}{3}} + \frac{y}{\frac{a}{4}} = 1 \quad \dots\dots\dots (1)$$

Area of the triangle with coordinates axes and  $\frac{x}{a} + \frac{y}{b} = 1$  is  $\frac{1}{2}|ab|$

$$\frac{1}{2} \left| \frac{a}{3} \cdot \frac{a}{4} \right| = 6 \quad \frac{a^2}{12} = 12$$

$$a^2 = 144 \quad a = \sqrt{144}$$

$$a = 12.$$

6. Transform the equation  $\frac{x}{a} + \frac{y}{b} = 1$  into the normal form when  $a > 0$  and  $b > 0$ . If the perpendicular distance of the straight line from the origin is  $P$ , deduce that

$$\frac{1}{P^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Sol:  $\frac{x}{a} + \frac{y}{b} = 1$

Divide with  $\sqrt{a^2 + b^2}$   $\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

$$\frac{\frac{x}{a}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} + \frac{\frac{y}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

$$x \frac{1}{a\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} + y \frac{1}{b\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

According to the problem  $P = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$   $\frac{1}{P} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Squaring on Both Sides

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

### The Straight Line

#### Exercise 3c

**Note :**

$L_1 = a_1x + b_1y + c_1 = 0$ ,  $L_2 = a_2x + b_2y + c_2 = 0$  then intersecting point of  $L_1, L_2$  is

$$\left( \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \right)$$

$L = ax + by + c = 0$  & two points  $A(x_1, y_1), B(x_2, y_2)$  then notation of  $L_{11} = ax_1 + by_1 + c_1$  and  $L_{22} = ax_2 + by_2 + c_2$

(a) If A & B are same side to  $L = 0$

$L_1$  &  $L_2$  has same sign

(b) If A & B are opposite side to  $L = 0$

$L_{11}$  &  $L_{22}$  has opposite signs.

(a) X-axis divides line segment  $\overline{AB}$  in the ratio  $= -y_1 : y_2$

(a) Y-axis divides the line segment  $\overline{AB}$  in the ratio  $= -x_1 : x_2$

$L_1 = 0, L_2 = 0, L_3 = 0$  are concurrent lines

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad \text{OR}$$

$$a_1(b_2c_3 - b_3c_2) - b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

### Questions

1. Find the ratio in which the straight line  $2x + 3y = 5$  divides the line segment joining the  $(0, 0)$  and  $(-2, 1)$ . State whether the points lie on the same side or neither side of the S.L.

**Sol:**  $L = 2x + 3y = 5$ ,  $A(0, 0), B(-2, 1)$

$$L_{11} = 2(0) + 3(0) - 5$$

$$L_{11} = -5 < 0$$

$$L_{22} = 2(-2) + 3(1) - 5 = -4 + 3 - 5 = -6$$

$$L_{22} = -6 < 0$$

$L = 0$  divides the line segment  $\overline{AB}$  in the ratio  $= -L_{11} : L_{22}$

Ratio

$$L_{11} < 0, L_{22} < 0 \text{ \& } L_{11}, L_{22} < 0$$

A, B are lie on the same side of the line  $L = 0$

- 2. Find the value of  $k$ , if the lines  $2x - 3y + k = 0$ ,  $3x - 4y - 13 = 0$  and  $8x - 11y - 33 = 0$  are concurrent.**

**Sol:**  $L_1 \quad 2x - 3y + k = 0, L_2 \quad 3x - 4y - 13 = 0, L_3 \quad 8x - 11y - 33 = 0$

$L_1, L_2, L_3$  are concurrent

$$\begin{vmatrix} 2 & 3 & k \\ 3 & 4 & 13 \\ 8 & 11 & 33 \end{vmatrix} = 0$$

$$2(132 - 143) + 3(-99 + 104) + k(-33 + 32) = 0$$

$$2(-11) + 3(5) + k(-1) = 0$$

$$k = -7.$$

Home work I<sub>5</sub> & II<sub>7</sub>

- 3. A triangle of area 24 sq. units is formed by a S.L. and the coordinate axes in the first quadrant. Find the equation of the S.L. if it passes through (3, 4).**

**Sol:**  $x = 0, y = 0, A(3, 4)$

intercept form of a S.L.  $\frac{x}{a} + \frac{y}{b} = 1$  .....(1)

Area of the triangle with coordinate axes & eqn (1) sq. units.

$$\frac{1}{2}|ab| = 24 \quad ab = 48 \quad b = \frac{48}{a} \quad \text{.....(2)}$$

But equation (1) is passing through A

$$\frac{3}{a} + \frac{4}{b} = 1 \quad \frac{3}{a} + \frac{4}{\frac{48}{a}} = 1$$

From equation (2)

$$\frac{3}{a} + \frac{4a}{48} = 1 \quad \frac{36 + a^2}{12a} = 1$$

$$36 + a^2 = 12a$$

$$a^2 - 12a + 36 = 0$$

$$(a - 6)^2 = 0$$

$$a = 6.$$

From equation (2)  $b = \frac{48}{6} = 8$

From (1) : Equation of required S.L.

$$\frac{x}{6} + \frac{y}{8} = 1 \quad \frac{4x + 3y}{24} = 1$$

$$4x + 3y = 24$$

$$4x + 3y - 24 = 0.$$

4. **If  $3a + 2b + 4c = 0$ , then show that the equation  $ax + by + c = 0$  represents a family of concurrent straight lines and find the point of concurrency**

**Sol:**  $3a + 2b + 4c = 0$

Divide with 4

$$\frac{3a}{4} + \frac{2b}{4} + \frac{4c}{4} = 0$$

$$a \cdot \frac{3}{4} + b \cdot \frac{1}{2} + c = 0 \quad \dots\dots(1)$$

Each member of the family of S.Lines given by  $ax + by + c = 0$  passes through the fixed point  $\left(\frac{3}{4}, \frac{1}{2}\right)$ . Hence the set of lines  $ax + by + c = 0$  for parametric values of  $a, b$  and  $c$  is

a family of concurrent lines and the point of concurrency is  $\left(\frac{3}{4}, \frac{1}{2}\right)$

H.W page No. 55 Example 4

5. **Find the point on the straight line  $3x + y + 4 = 0$  which is equidistant from the points  $(-5, 6)$  and  $(3, 2)$ .**

**Sol:** A(-5, 6), B(3, 2), L  $3x + y + 4 = 0$

Let P(a, b) be the required print

$$AP = BP$$

$$\sqrt{(a + 5)^2 + (b - 6)^2} = \sqrt{(a - 3)^2 + (b - 2)^2}$$

Squaring on B.S.

$$a^2 + 25 + 10a + b^2 + 36 - 12b = a^2 + 9 - 6a + b^2 + 4 - 4b$$

$$10a - 12b + 61 + 6a + 4b - 13 = 0$$

$$16a - 8b + 48 = 0$$

$$\text{Divide with } 2a - b + 6 = 0 \quad \dots\dots(1)$$

But P is on L = 0

$$3a + b + 4 = 0 \quad \dots\dots(2)$$

Now

$$2a - b + 4 = 0$$

$$\underline{3a + b + 4 = 0}$$

$$5a + 10 = 0 \quad 5a = -10$$

$$a = -2$$

$$\text{From } 2(-2) - b + 6 = 0 \quad -4 + 6 = b$$

$$b = 2$$

Required point P = (-2, 2).

6. A straight line through P(3, 4) makes an angle of  $60^\circ$  with the positive direction of the X-axis. Find the coordinates of the points on the line which are 5 units away from P.

Sol: P(3, 4),  $r = 5$ ,  $\theta = 60^\circ$

Parametric form of a point

$$(x, y) = (x_1 \pm r \cos \theta, y_1 \pm r \sin \theta)$$

The point on the line which is at a distance of 5 units from P

$$(x, y) = (3 \pm 5 \cos 60^\circ, 4 \pm 5 \sin 60^\circ)$$

$$3 + 5 \cdot \frac{1}{2}, 4 + 5 \cdot \frac{\sqrt{3}}{2}$$

$$\frac{6 + 5}{2}, \frac{8 + 5\sqrt{3}}{2}$$

$$\frac{6 - 5}{2}, \frac{8 - 5\sqrt{3}}{2}, \frac{6 + 5}{2}, \frac{8 - 5\sqrt{3}}{2}$$

$$\text{Required points } \frac{11}{2}, \frac{8 + 5\sqrt{3}}{2}, \frac{1}{2}, \frac{8 - 5\sqrt{3}}{2}$$

7. A S.L through Q( $\sqrt{3}$ , 2) makes an angle  $30^\circ$  with the positive direction of the X-axis. If the S.L. intersects the line  $\sqrt{3}x - 4y - 8 = 0$  at P, find the distance PQ

Sol:  $\sqrt{3}x - 4y - 8 = 0 \quad \dots\dots(1)$



$$Q(\sqrt{3}, 2), \quad \frac{180}{6} = 30 \quad m = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

Equation of a S.L. with slope  $\frac{1}{\sqrt{3}}$  and passing through  $Q(\sqrt{3}, 2)$

$$y - 2 = \frac{1}{\sqrt{3}}(x - \sqrt{3})$$

$$\sqrt{3}(y - 2) = x - \sqrt{3}$$

$$\sqrt{3}y - 2\sqrt{3} = x - \sqrt{3}$$

$$x - \sqrt{3}y + \sqrt{3} = 0 \quad \dots\dots(2)$$

For P:  $\sqrt{3}$

$$\sqrt{3}x - 4y + 8 = 0$$

$$\sqrt{3}x - 3y - 3 = 0$$

$$-y + 5 = 0 \quad y = 5.$$

from  $\sqrt{3}x - 4(5) + 8 = 0 \quad \sqrt{3}x - 12 = 0 \quad x = \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$

P  $(4\sqrt{3}, 5)$

$$PQ = \sqrt{(\sqrt{3} - 4\sqrt{3})^2 + (2 - 5)^2}$$

$$= \sqrt{3\sqrt{3}^2 + (-3)^2} = \sqrt{9(3) + 9} = \sqrt{36}$$

$$PQ = 6.$$

H.W. Page 54 : Example 3

### The Straight Line

#### Exercise 3d

**Note :**

Let  $\theta$  be the angle between the line  $L_1$  &  $L_2$  then  $\cos \theta = \frac{|a_1a_2 + b_1b_2|}{\sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}}$

$$L_1 \perp L_2 \quad a_1a_2 + b_1b_2 = 0 \quad \text{OR} \quad m_1.m_2 = -1$$

Perpendicular distance from  $P(x_1, y_1)$  to  $L: ax + by + c = 0$  is  $(d) \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$

Distance between the parallel lines  $L_1: ax + by + c_1 = 0$  &  $L_2: ax + by + c_2 = 0$

$$\text{is } \left| \frac{c_1 - c_2}{\sqrt{a^2 + b^2}} \right|$$

### Questions

1. Find the value of  $k$ , if the angle between the straight lines  $4x - y + 7 = 0$  and  $kx - 5y - 9 = 0$  is  $45^\circ$

**Sol:**  $L_1: 4x - y + 7 = 0$ ,  $L_2: kx - 5y - 9 = 0$ ,  $45^\circ$

$$\cos \theta = \frac{|a_1 a_2 + b_1 b_2|}{\sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}}$$

$$\cos 45^\circ = \left| \frac{4k + (-1)(-5)}{\sqrt{4^2 + (-1)^2} \sqrt{k^2 + (-5)^2}} \right|$$

$$\frac{1}{\sqrt{2}} = \frac{4k + 5}{\sqrt{16 + 1} \sqrt{k^2 + 25}}$$

$$\frac{1}{2} = \frac{16k^2 + 25 + 40k}{17(k^2 + 25)}$$

$$17k^2 + 425 + 32k^2 + 50 = 80k$$

$$15k^2 + 80k + 375 = 0$$

$$3k^2 + 16k + 75 = 0$$

$$3k^2 + 25k + 9k + 75 = 0$$

$$k(3k + 25) + 3(3k + 25) = 0$$

$$(3k + 25)(k + 3) = 0$$

$$3k + 25 = 0 \text{ OR } k + 3 = 0$$

$$3k = -25 \text{ OR } k = -3$$

$$k = -3 \text{ OR } \frac{25}{3}$$

2. Find the value of  $k$ , if the straight lines  $y - 3kx + 4 = 0$  and  $(2k - 1)x - (8k - 1)y - 6 = 0$  are perpendicular

Sol:  $L_1 \quad y - 3kx + 4 = 0 \quad L_2 \quad 3kx - y - 4 = 0 \dots\dots\dots (1)$

$L_2 \quad (2k - 1)x - (8k - 1)y - 6 = 0 \dots\dots\dots (2)$

$L_1 \quad L_2 \quad a_1a_2 \quad b_1b_2 \quad 0$

$3k(2k - 1) - (1)[(8k - 1)] = 0$

$6k^2 - 3k - 8k + 1 = 0$

$6k^2 - 11k + 1 = 0$

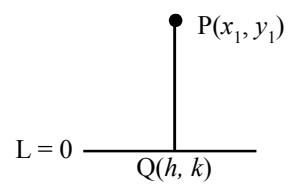
$6k^2 - 6k - 5k + 1 = 0$

$6k(k - 1) - 5(k - 1) = 0$

$(k - 1)(6k - 5) = 0$

$k - 1 = 0 \text{ OR } 6k - 5 = 0$

$k = 1 \text{ OR } k = \frac{5}{6}$



3. Find the length of the perpendicular distance from the point  $(-2, -3)$  to the line  $5x - 2y + 4 = 0$ .

Sol:  $L \quad 5x - 2y + 4 = 0, \quad P(-2, -3)$

(d)  $\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$

Distance between parallel line (1) & (2) (d)  $\left| \frac{5(-2) - 2(-3) + 4}{\sqrt{(5)^2 + (-2)^2}} \right| = \left| \frac{10 - 6 + 4}{\sqrt{25 + 4}} \right|$

$d = 0.$

Home Work I<sub>9, 10</sub>

4. Find the distance between parallel lines  $3x + 4y - 3 = 0, \quad 6x + 8y - 1 = 0$

Sol:  $3x + 4y - 3 = 0 \quad 6x + 8y - 1 = 0 \dots\dots (2)$

$2 \times (1) \quad 6x + 8y - 6 = 0 \dots\dots (1)$

distance between parallel lines  $\left| \frac{6 - (-6)}{\sqrt{(6)^2 + (8)^2}} \right|$

$\frac{|6 - (-6)|}{\sqrt{36 + 64}} = \frac{|12|}{\sqrt{100}} = \frac{12}{10} = \frac{6}{5}$

$d = \frac{6}{5}$

5. If  $Q(h, k)$  is the foot of the perpendicular from  $P(x_1, y_1)$  on the straight line  $ax + by + c = 0$ , then

$$(h - x_1) : a = (k - y_1) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)$$

**Sol:** L  $ax + by + c = 0$ ,  $P(x_1, y_1)$ ,  $Q(h, k)$

$$\text{Slope of L } m_1 = \frac{a}{b}$$

$$\text{Slope of PQ } (m_2) = \frac{k - y_1}{h - x_1}$$

$$L \perp PQ \quad m_1 m_2 = -1$$

$$\frac{a}{b} \cdot \frac{k - y_1}{h - x_1} = -1$$

$$\frac{k - y_1}{h - x_1} = \frac{b}{a} \cdot \frac{k - y_1}{h - x_1} = \frac{h - x_1}{a}$$

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} \quad \text{say}$$

$$\frac{h - x_1}{a} = \frac{k - y_1}{b}$$

$$h - x_1 = a \cdot \frac{k - y_1}{b}$$

$$h - x_1 = a \cdot \frac{k - y_1}{b}$$

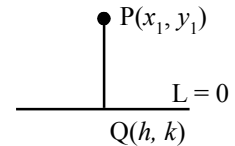
But  $Q$  is on  $L = 0$

$$a(x_1 - a) + b(y_1 - b) + c = 0$$

$$ax_1 - a^2 + by_1 - b^2 + c = 0$$

$$(a^2 - b^2) + ax_1 - by_1 + c = \frac{(ax_1 - by_1 - c)}{a^2 - b^2}$$

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{(ax_1 - by_1 - c)}{a^2 - b^2}$$



6. Find the foot of the perpendicular from  $(-1, 3)$  on the straight line  $5x - y - 18 = 0$

**Sol:** Let  $Q(h, k)$  be the foot of the perpendicular from  $(-1, 3)$  to  $5x - y - 18 = 0$

$$\frac{h - (-1)}{5} = \frac{k - 3}{-1} = \frac{(5(-1) - 3 - 18)}{(5)^2 - (-1)^2}$$

$$\frac{h + 1}{5} = \frac{k - 3}{-1} = \frac{(-5 - 3 - 18)}{(5)^2 - (1)^2}$$

$$\frac{h-1}{5} = \frac{k-3}{1} = \frac{26}{25-1}$$

$$\frac{h-1}{5} = \frac{k-3}{1} = \frac{26}{26} = 1$$

$$\frac{h-1}{5} = 1 \qquad \frac{k-3}{1} = 1$$

$$h-1 = 5 \qquad k-3 = -1$$

$$h = 5 + 1 \qquad k = -1 + 3$$

$$h = 4 \qquad k = 2$$

$$(h, k) = (4, 2)$$

Home Work II<sub>5</sub>

7. If  $Q(h, k)$  is the image of the point  $P(x_1, y_1)$  w.r.t. the straight line  $ax + by + c = 0$  then  $(h - x_1) : a = (k - y_1) : b = -2(ax_1 + by_1 + c) : (a^2 + b^2)$

Sol: L  $ax + by + c = 0$  P( $x_1, y_1$ ) Q( $h, k$ )

$$\text{Slope of L} = 0 \ (m_1) = \frac{a}{b}$$

$$\text{Slope of PQ} \ (m_2) = \frac{k - y_1}{h - x_1}$$

$$PQ \perp L \quad m_1 m_2 = -1$$

$$\frac{a}{b} \cdot \frac{k - y_1}{h - x_1} = -1$$

$$\frac{k - y_1}{h - x_1} = \frac{b}{a} = \frac{k - y_1}{b} = \frac{h - x_1}{a}$$

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} \quad \text{let .....(1)}$$

$$\frac{h - x_1}{a} = \frac{k - y_1}{b}$$

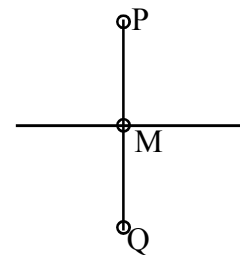
$$h - x_1 = a \cdot \frac{k - y_1}{b}$$

$$h - x_1 = a \cdot \frac{k - y_1}{b} \quad \text{.....(2)}$$

Since M is the midpoint of P & Q

$$M \left( \frac{x_1 + h}{2}, \frac{y_1 + k}{2} \right)$$

It is on L = 0



$$a \frac{x_1 - h}{2} + b \frac{y_1 - k}{2} + c = 0$$

$$\frac{ax_1 - ah + by_1 - bk + 2c}{2} = 0$$

$$ax_1 - a(x_1 - a) + by_1 - b(y_1 - b) + 2c = 0$$

$$ax_1 - ax_1 + a^2 + by_1 - by_1 + b^2 + 2c = 0$$

$$(a^2 + b^2) + 2ax_1 + 2by_1 + 2c = 0$$

$$\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

From (1)

$$\frac{h - x_1}{a} + \frac{k - y_1}{b} = \frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

**8. Find the image of (1, -2) w.r.t. the straight line  $2x - 3y + 5 = 0$ .**

**Sol:**  $(h, k)$  is the image of  $(1, -2)$  w.r.t the line  $2x - 3y + 5 = 0$

$$\frac{h - 1}{2} + \frac{k - 2}{3} = \frac{2(2(1) - 3(-2) + 5)}{(2)^2 + (-3)^2}$$

$$\frac{h - 1}{2} + \frac{k - 2}{3} = \frac{2(2 - 6 + 5)}{4 + 9} = \frac{2(1)}{13} = \frac{2}{13}$$

$$\frac{h - 1}{2} + 2 = \frac{k - 2}{3} + 2$$

$$h - 1 = 4 \quad k - 2 = 6$$

$$h = 5 \quad k = 8$$

$$(h, k) = (5, 8)$$

Home Work II<sub>8</sub>

**9. Find the equations of the straight lines passing through the point  $(-3, 2)$  and making an angle of  $45^\circ$  with the straight line  $3x - y + 4 = 0$**

**Sol:**  $P(-3, 2)$ ,  $L: 3x - y + 4 = 0$ ,  $\theta = 45^\circ$

Let  $m$  be the slope of a line which is passing through  $P$

$$y - 2 = m(x + 3) \quad \dots(*)$$

$$y - 2 = mx + 3m$$

$$mx - y + (2 + 3m) = 0 \quad \dots(1)$$

According to the problem,  $45^\circ$  is the angle between

$L = 0$  &

$$\cos 45^\circ = \frac{3(m-1)(-1)}{\sqrt{(3)^2 - (1)^2 - m^2 - (-1)^2}} = \frac{1}{\sqrt{2}} = \frac{3m-1}{\sqrt{(9-1)(m^2-1)}}$$

Squaring on B.S.

$$\frac{1}{2} = \frac{9m^2 - 1 - 6m}{10(m^2 - 1)}$$

$$10(m^2 - 1) = 2(9m^2 - 1 - 6m)$$

$$10m^2 - 10 = 18m^2 - 2 - 12m$$

$$8m^2 - 12m + 8 = 0$$

$$2m^2 - 3m + 2 = 0$$

$$2m^2 - 4m + 1m + 2 = 0$$

$$2m(m - 2) + 1(m - 2) = 0$$

$$(m - 2)(2m + 1) = 0$$

$$m + 2 = 0 \quad \text{OR} \quad 2m - 1 = 0$$

$$m = -2 \quad \text{OR} \quad m = \frac{1}{2}$$

Case (i):

$m = -2$  then, from equation (\*)

$$y - 2 = 2(x - 3)$$

$$y - 2 = 2x - 6$$

$$2x - y + 4 = 0$$

Case (ii):

$m = \frac{1}{2}$  then, from equation (\*)

$$y - 2 = \frac{1}{2}(x - 3)$$

$$2y - 4 = x - 3$$

$$x - 2y + 7 = 0$$

Equation of required s. lines

$$2x + 4y + 4 = 0,$$

$$x - 2y + 7 = 0$$

**10. Find the angles of the triangle whose sides are  $x + y - 4 = 0$ ,  $2x + y - 6 = 0$  and  $5x + 3y - 15 = 0$ .**

**Sol:**  $L_1 \quad x + y - 4 = 0$

$L_2 \quad 2x + y - 6 = 0$

$$L_3 \quad 5x + 3y - 15 = 0$$

$$\cos A = \frac{|1(2) - 1(1)|}{\sqrt{(1^2 - 1^2)(2^2 - 1^2)}} = \frac{|2 - 1|}{\sqrt{(1 - 1)(4 - 1)}} = \frac{3}{\sqrt{10}}$$

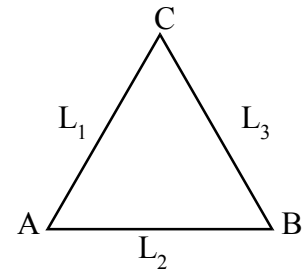
$$A = \cos^{-1} \frac{3}{\sqrt{10}} = 18.45^\circ$$

$$\cos B = \frac{|2(5) - 1(3)|}{\sqrt{(2^2 - 1^2)(5^2 - 3^2)}} = \frac{|10 - 3|}{\sqrt{(4 - 1)(25 - 9)}} = \frac{13}{\sqrt{170}}$$

$$B = \cos^{-1} \frac{13}{\sqrt{170}} = 4.43^\circ$$

$$\cos C = \frac{|5(1) - 3(1)|}{\sqrt{(5^2 - 3^2)(1^2 - 1^2)}} = \frac{|5 - 3|}{\sqrt{(25 - 9)(1 - 1)}} = \frac{8}{\sqrt{68}} = \frac{8}{2\sqrt{17}} = \frac{4}{\sqrt{17}}$$

$$C = \cos^{-1} \frac{4}{\sqrt{17}} = 14.02^\circ$$



$A + B + C = 36.9^\circ$  which is in correct.

$\therefore$  Sum of the angles in triangle is  $180^\circ$ .

$$A = \cos^{-1} \frac{3}{\sqrt{10}}$$

- 11. Find the equations of the straight lines passing through the point of intersection of the lines  $3x + 2y + 4 = 0$ ,  $2x + 5y = 1$  and whose distance from  $(2, -1)$  is 2.**

**Sol:**  $L_1 \quad 3x + 2y + 4 = 0$ ,  $L_2 \quad 2x + 5y = 1$ ,  $A = (2, -1)$ ,  $d = 2$

For the point of intersection of  $L_1$  &  $L_2$

$$2L_1 - 3L_2$$

$$6x + 4y + 8 = 0$$

$$6x + 15y - 3 = 0$$

$$-11y + 11 = 0 \quad 11y = 11 \quad y = 1$$

$$\text{From } L_1 \quad 3x + 2(1) + 4 = 0 \quad 3x = -6 \quad x = -2$$

$P(2, -1)$

Let  $m$  be the slope of a line which is passing through  $P$

$$y - 1 = m(x + 2) \quad \dots(*)$$

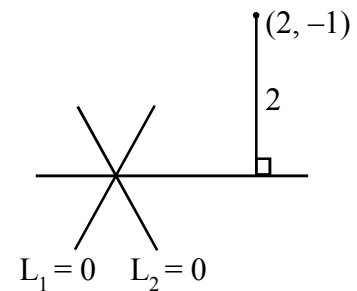
$$y - 1 = mx + 2m$$

$$mx - y + (1 + 2m) = 0 \quad \dots(1)$$

Given that perpendicular distance from

$P$  to equation (1) = 2

$$\left| \frac{m(2) - (-1) + 1 + 2m}{\sqrt{m^2 + (-1)^2}} \right| = 2$$





$$\frac{2m - 1}{\sqrt{m^2 - 1}} = \frac{2m}{\sqrt{1 - m^2}}$$

$$2(2m - 1) = 2\sqrt{1 - m^2}$$

Squaring on both sides

$$4m^2 - 4m + 1 = 4(1 - m^2)$$

$$3m^2 - 4m + 0 = 0$$

$$m(3m - 4) = 0$$

$$m = 0 \text{ OR } m = \frac{4}{3}$$

Case (i):  $m = 0$  then, from (\*)

$$y - 1 = 0(x - 3)$$

$$y = 1$$

Case (ii):  $m = \frac{4}{3}$  then, from (\*)

$$y - 1 = \frac{4}{3}(x - 2)$$

$$3y - 3 = 4x - 8$$

$$4x - 3y - 5 = 0$$

Equation of required straight lines  $y = 1, 4x - 3y - 5 = 0$

**12. Find the equation of the line perpendicular to the line  $3x + 4y + 6 = 0$  and making intercept  $-4$  on the X-axis.**

**Sol:**  $L_1: 3x + 4y + 6 = 0$ , X-intercept  $= -4$

Equation of any line which is perpendicular to  $L = 0$

$$4x - 3y + k = 0 \quad \dots(*)$$

$$4x - 3y = -k$$

$$\frac{4x}{(-k)} = \frac{3y}{(-k)} = \frac{k}{(-k)}$$

$$\frac{x}{\frac{k}{4}} = \frac{y}{\frac{k}{3}} = 1 \quad \dots(1)$$

X-intercept of (1)  $= \frac{k}{4}$

According to the problem  $\frac{k}{4} = -4 \implies k = -16$

Equation of required straight line  $4x - 3y + 16 = 0$

## The Straight Line

### Exercise 3e

**Note :**

The medians of a triangle are concurrent. The concurrent point of medians is the centroid.  
The altitudes of a triangle are concurrent. The concurrent point of the altitudes is ortho-center.

The internal bisectors of the angles of a triangle are concurrent. The concurrent point is called incenter.

4. The perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrency is the 'circumcenter' of the triangle.

### Questions

- 1. Find the equation of the straight line passing through the origin and also through the point of intersection of the lines  $2x - y + 5 = 0$  and  $x + y + 1 = 0$ .**

**Sol:**  $O(0, 0)$ ,  $L_1 \quad 2x - y + 5 = 0$ ,  $L_2 \quad x + y + 1 = 0$

$$\text{From } L_1 + L_2 \quad 2x - y + 5 = 0$$

$$\quad \quad \quad \underline{x + y + 1 = 0}$$

$$\quad \quad \quad 3x + 6 = 0 \quad 3x = -6 \quad x = -2$$

$$\text{From } L_2 : 2(-2) - y + 5 = 0 \quad 1 - y = 0 \quad y = 1$$

intersecting point of  $L_1$  &  $L_2$   $A(-2, 1)$

Equation of required straight line passing through  $O$  &  $A$

$$y - 0 = \frac{1 - 0}{-2 - 0} (x - 0)$$

$$2(y) = 1(x)$$

$$x + 2y = 0$$

- 2. Find the equation of the straight line parallel to the line  $3x + 4y = 7$  and passing through the point of intersection of the lines  $x - 2y - 3 = 0$  and  $x + 3y - 6 = 0$**

**Sol:**  $L_1 \quad 3x + 4y = 7$ ,  $L_2 \quad x - 2y - 3 = 0$ ,  $L_3 \quad x + 3y - 6 = 0$

To find the point intersection of  $L_2, L_3$

$$x - 2y - 3 = 0$$

$$\underline{x + 3y - 6 = 0}$$

$$- \quad - \quad +$$

$$-5y + 3 = 0$$

$$5y = 3 \quad y = \frac{3}{5}$$

$$\text{From } L_2 \quad x - 2\left(\frac{3}{5}\right) - 3 = 0$$

$$x = \frac{6}{5}, y = 3 - \frac{6}{5} = \frac{15}{5} - \frac{6}{5} = \frac{9}{5}$$

Equation of any line parallel to  $L_1$  is  $3x + 4y = k$

Equation of any line parallel to  $L_1 = 0$  is  $3x + 4y + k = 0$

But it is passing through A

$$3 \left(\frac{6}{5}\right) + 4 \left(\frac{9}{5}\right) + k = 0$$

$$k = -\frac{63}{5} - \frac{36}{5} = -\frac{99}{5} = -19.8$$

Equation of required straight line  $3x + 4y - 19.8 = 0$

3. Find the equation of the straight line making non-zero equal intercepts on the coordinate axes and passing through the point of intersection of the lines  $2x - 5y + 1 = 0$  and  $x - 3y - 4 = 0$

Sol:  $L_1: 2x - 5y + 1 = 0, L_2: x - 3y - 4 = 0$

To find the point intersection of  $L_1, L_2$

$$\begin{array}{r} 2x - 5y + 1 = 0 \\ 2x - 6y - 8 = 0 \\ \hline - \quad + \quad + \\ y + 9 = 0 \quad y = -9 \end{array}$$

$$\begin{array}{l} \text{From } L_1 \quad 2x - 5(-9) + 1 = 0 \\ \quad \quad \quad 2x + 45 + 1 = 0 \\ \quad \quad \quad 2x = -46 \\ \quad \quad \quad x = -23 \end{array}$$

A = (-23, -9)

Since equal intercepts on the coordinate axes

$$a = b$$

Equation of S.L. in intercept form

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \left( \frac{x}{a} + \frac{y}{a} = 1 \quad \because a = b \right)$$

But it is passing through A

$$-\frac{23}{a} - \frac{9}{a} = 1$$

$$-32 = a$$

Equation of a required straight line  $x + y = -32 \quad x + y + 32 = 0$ .

4. Find the length of the perpendicular drawn from the point of intersection of the lines  $3x + 2y + 4 = 0$  and  $2x + 5y - 1 = 0$  to the straight line  $7x + 24y - 15 = 0$

Sol:  $L_1$   $3x + 2y + 4 = 0$ ,  $L_2$   $2x + 5y - 1 = 0$ ,  $L_3$   $7x + 24y - 15 = 0$

Point of intersection of  $L_1$  &  $L_2$

$$\frac{x}{2(1) - 5(4)} \quad \frac{y}{4(2) - (1)(3)} \quad \frac{1}{3(5) - 2(2)}$$

$$\frac{x}{2 - 20} \quad \frac{y}{8 - 3} \quad \frac{1}{15 - 4}$$

$$\frac{x}{-18} \quad \frac{y}{5} \quad \frac{1}{11}$$

$$A = (-2, 1)$$

Perpendicular distance from A to  $L_3 = 0$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$\frac{|7(-2) + 24(1) - 15|}{\sqrt{(7)^2 + (24)^2}} = \frac{|-14 + 24 - 15|}{\sqrt{49 + 576}}$$

$$\frac{|-5|}{\sqrt{625}} = \frac{5}{25}$$

$$d = \frac{1}{5}$$

5. The base of an equilateral triangle is  $x + y - 2 = 0$  and the opposite vertex is  $(2, -1)$ . Find the equations of the remaining sides.

Sol:  $L$   $x + y - 2 = 0$ ,  $A(2, -1)$

Let 'm' be the slope of a line which is passing through A.

Equation of a line with slope  $m$  and passing through A is

equation of the line passing through A and intersecting  $L = 0$  at B

$$y + 1 = m(x - 2) \quad \dots(*)$$

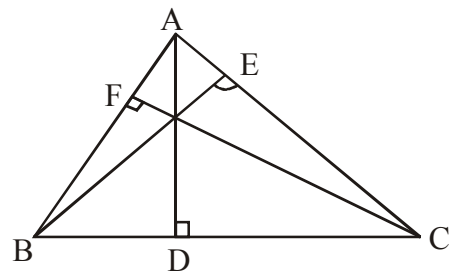
$$y + 1 = mx - 2m$$

$$mx - y - (1 + 2m) = 0 \quad \dots(1)$$

Since  $\triangle ABC$  is equilateral triangle

$$\angle A = \angle B = \angle C$$

$$\text{Angle between } L \text{ and } L = 0, (1)$$



$$\cos 60^\circ = \frac{1(m) - 1(-1)}{\sqrt{((1)^2 - (1)^2)(m^2 - (-1)^2)}}$$

Squaring on both sides

$$\frac{1}{2} = \frac{m - 1}{\sqrt{(1 - 1)(1 - m^2)}}$$

$$\frac{x}{2} = \frac{y}{8} = \frac{1}{15} = \frac{4}{4}$$

$$\frac{1}{4} = \frac{m^2 - 1 - 2m}{2(1 - m^2)}$$

$$(1 - m^2) - 2m^2 - 2 - 4m$$

$$2m^2 - 2 - 4m - 1 - m^2 = 0$$

$$m^2 - 4m - 1 = 0$$

$$m = \frac{(4) \pm \sqrt{(4)^2 - 4(1)(-1)}}{2(1)}$$

$$\frac{4 \pm \sqrt{16 - 4}}{2} = \frac{4 \pm \sqrt{12}}{2}$$

$$\frac{4 \pm 2\sqrt{3}}{2} = \frac{2(2 \pm \sqrt{3})}{2}$$

$$m = 2 \pm \sqrt{3}$$

From (\*) : Equation of required S.L.  $y - 1 = (2 \pm \sqrt{3})(x - 2)$

Home work II<sub>2</sub>

**6. Find the orthocenter of the triangle whose vertices are (-5, -7), (13, 2) and (-5, 6)**

**Sol:** A(-5, -7), B(13, 2), C(-5, 6)

$$\text{Slope of BC} = \frac{6 - 2}{-5 - 13} = \frac{4}{-18} = -\frac{2}{9}$$

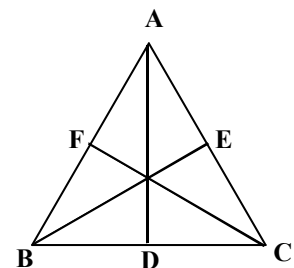
Let AD be the perpendicular drawn from A to BC

Altitude drawn from CF to AB

Let BE = be the perpendicular drawn from B To  $\overline{AC}$

AD  $\perp$  BC

$$\text{Slope of AD} = \frac{1}{m} = \frac{1}{-\frac{2}{9}} = -\frac{9}{2}$$



Equation of a line with Slope  $\frac{9}{2}$  and passing through A is

$$y - 7 = \frac{9}{2}(x - 5)$$

$$2y - 14 = 9x - 45$$

$$9x - 2y - 31 = 0$$

$$\text{Slope of AC} = \frac{6 - 7}{5 - (-5)} = \frac{13}{10}$$

BE  $\perp$  AC Slope of BE = 0

$$\text{Equation of BE} \quad y - 2 = 0(x - 13)$$

$$y - 2 = 0$$

$$y = 2 \quad \dots(2)$$

Intersecting point of (1) & (2) is our required orthocenter

$$\text{From (1) \& (2)} \quad 9x - 2(2) + 31 = 0$$

$$9x + 27 = 0 \quad 9x = -27 \quad 9x = -27$$

$$O = (-3, 2)$$

7. If the equations of the sides of a triangle are  $7x + y - 10 = 0$ ,  $x - 2y + 5 = 0$  and  $x + y + 2 = 0$ , find the orthocenter of the triangle.

**Sol:**  $L_1: 7x + y - 10 = 0$ ,  $L_2: x - 2y + 5 = 0$ ,  $L_3: x + y + 2 = 0$

For A  $2L_1 + L_2$

$$14x + 2y - 20 = 0$$

$$\underline{x - 2y + 5 = 0}$$

$$15x - 15 = 0 \quad x = 1.$$

$$\text{From } L_1 \quad 7(1) + y - 10 = 0 \quad y - 3 = 0 \quad y = 3 \quad A = (1, 3)$$

For B  $L_1 - L_3$

$$7x + y - 10 = 0$$

$$\underline{x + y + 2 = 0}$$

- - -

$$6x - 12 = 0 \quad x = 2.$$

$$\text{from } L_1 \quad 7(2) + y - 10 = 0 \quad y + 4 = 0 \quad y = -4 \quad B = (2, -4)$$

For C  $L_2 - L_3$

$$x - 2y + 5 = 0$$

$$x + y + 2 = 0$$

$$-3y + 3 = 0 \quad y = 1.$$

$$L_2 \text{ from } x - 2(1) + 5 = 0 \quad x + 3 = 0 \quad x = -3$$

$$C = (-3, 1)$$

Slope of BC  $\frac{1}{(1)} = 1$

AD  $\perp$  BC Slope of AD  $\frac{1}{(1)} = 1$

Equation of AD  $y - 3 = 1(x - 1)$   
 $y - 3 = (x - 1)$   
 $x - y + 2 = 0 \dots(1)$

Slope of AC  $\frac{1}{(2)} = \frac{1}{2}$

BE  $\perp$  AC Slope of BE  $\frac{1}{(1/2)} = 2$

Equation of BE  $y + 4 = -2(x - 2)$   
 $y + 4 = -2x + 4$   
 $2x + y = 0 \dots(2)$

Intersecting point of & is our required orthocenter (1

$$\begin{array}{r} x - y + 2 = 0 \\ 2x + y = 0 \\ \hline 3x + 2 = 0 \quad x = -2/3 \end{array}$$

From  $\frac{2}{3} y + 2 = 0 \quad \frac{2}{3} y = -2 \quad y = -\frac{4}{3}$

required orthocentre O =  $\left(-\frac{2}{3}, -\frac{4}{3}\right)$

Home Work I<sub>3</sub>, II<sub>4</sub> & III<sub>1</sub>

**8. Find the circumcenter of the triangle whose vertices are (-2, 3), (2, -1) and (4, 0)**

**Sol:** A(-2, 3), B(2, -1), C(4, 0)

Let S = (a, b) be the circumcenter of the ABC

We know that SA = SB = SC

From : SA = SB

$$\begin{aligned} \sqrt{(2 - a)^2 + (3 - b)^2} &= \sqrt{(2 - a)^2 + (-1 - b)^2} \\ 4 - a^2 - 4a + 9 - b^2 - 6b &= 4 - a^2 - 4a + 1 - b^2 - 2b \\ 4a - 6b - 13 &= 4a - 2b - 5 - 0 \\ 8a - 8b - 8 &= 0 \\ \text{Divide with 8} \quad a - b + 1 &= 0 \dots\dots(1) \\ \text{SA} &= \text{SB} \end{aligned}$$

$$\sqrt{(2-a)^2 + (3-b)^2} = \sqrt{(4-a)^2 + (0-b)^2}$$

Squaring on B.S

$$4a^2 - 4a + 9b^2 - 6b + 16a^2 - 8a + b^2$$

$$4a - 6b + 13 - 8a + 16 - 0$$

$$12a - 6b + 3 - 0$$

$$4a - 2b + 1 - 0 \quad \dots\dots(2)$$

Intersecting point of (1) & (2) is our required circumcenter

$$4a - 4b + 4 = 0$$

$$4a - 2b - 1 = 0$$

$$- \quad + \quad +$$

$$-2b + 5 = 0 \quad b = 5/2$$

$$\text{From (1) } a = \frac{5}{2}, b = 1 \quad \text{(2) } a = \frac{5}{2}, b = 1 \quad \text{(3) } a = \frac{5}{2}, b = \frac{3}{2}$$

$$\text{required circumcentre } S = \left( \frac{3}{2}, \frac{5}{2} \right)$$

Home Work (1) page 73 Example (2) 4  $\Pi_{5ii}$  (3)  $\text{III}_2$  (4)  $\text{I}_{13}$

**9. If  $p$  and  $q$  are the lengths of the perpendiculars from the origin to the straight lines**

$$x \sec \theta + y \operatorname{cosec} \theta = a \quad \text{and} \quad x \cos \theta - y \sin \theta = a \cos 2\theta \quad \text{Prove that } 4p^2 + q^2 = a^2.$$

**Sol:**  $L_1: x \sec \theta + y \operatorname{cosec} \theta - a = 0$ ,  $L_2: x \cos \theta - y \sin \theta - a \cos 2\theta = 0$

$p$  = perpendicular distance from  $(0, 0)$  to  $L_1 = 0$

$$\frac{\left| \frac{(0) \sec \theta + (0) \operatorname{cosec} \theta - a}{\sqrt{(\sec \theta)^2 + (\operatorname{cosec} \theta)^2}} \right|}{\frac{a}{\sqrt{\frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta}}}} = \frac{a}{\sqrt{(\sin^2 \theta + \cos^2 \theta)}} = \frac{a}{\sqrt{1}} = \frac{a}{1} = a$$

$$p = a \sin \theta \cos \theta$$

Now  $q$  = perpendicular distance from  $(0, 0)$  to  $L_2 = 0$

$$\frac{\left| \frac{(0) \cos \theta - (0) \sin \theta - a \cos 2\theta}{\sqrt{(\cos \theta)^2 + (\sin \theta)^2}} \right|}{\frac{a \cos 2\theta}{\sqrt{\cos^2 \theta + \sin^2 \theta}}} = \frac{a \cos 2\theta}{\sqrt{1}} = a \cos 2\theta$$



$$q = a \cos 2$$

$$\begin{aligned} \text{Now } 4p^2 + q^2 &= 4(a \sin \theta)^2 + (a \cos 2\theta)^2 \\ &= a^2(2 \sin \theta)^2 + a^2 \cos^2 2\theta \\ &= a^2[\sin^2 2\theta + \cos^2 2\theta] \\ &= a^2(1) \quad [\because \sin^2 \theta + \cos^2 \theta = 1] \\ 4p^2 + q^2 &= a^2 \end{aligned}$$

## Pair of Straight Lines

### Key Concepts

If  $a, b, h$  are real numbers, not all zero, then  $H \quad ax^2 + 2hxy + by^2 = 0$  is called a homogeneous equation of second degree in  $x$  and  $y$

$S \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  is called a general equation of second degree in  $x$  and  $y$ .

If  $a, b, h$  are not all zero, then the equation  $H \quad ax^2 + 2hxy + by^2 = 0$  represents a pair of straight lines  $h^2 = ab$

Let the equation  $ax^2 + 2hxy + by^2 = 0$  represent a pair of straight lines. Then the angle between the lines is given by

$$\cos \theta = \frac{|a - b|}{\sqrt{(a + b)^2 - 4h^2}}$$

- (i) If  $h^2 = ab$  then the lines are coincide
- (ii) If  $a + b = 0$  then the lines are perpendicular

If the second degree equation  $S \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  in the two variables  $x$  and  $y$  represents a pair of straight lines

- (i)  $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
- (ii)  $h^2 = ab, g^2 = ac, f^2 = bc$

### Long Answer Questions (7 Marks)

1. If  $\theta$  is an angle between the lines represented by  $ax^2 + 2hxy + by^2 = 0$  then prove that

$$\cos \theta = \frac{|a - b|}{\sqrt{(a + b)^2 - 4h^2}}$$

**Sol:** Let  $ax^2 + 2hxy + by^2 = 0$  represents the lines

$$l_1x + m_1y = 0$$

$$l_2x + m_2y = 0$$

$$ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$$

$$ax^2 + 2hxy + by^2 = l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$$

by comparing like term coefficients on both sides

$$l_1l_2 = a, \quad l_1m_2 + l_2m_1 = 2h, \quad m_1m_2 = b$$

If  $\theta$  is an angle between (1) and (2) then

$$\cos \theta = \frac{|l_1l_2 + m_1m_2|}{\sqrt{l_1^2 + m_1^2} \sqrt{l_2^2 + m_2^2}}$$

$$= \frac{|l_1l_2 + m_1m_2|}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}}$$

$$= \frac{|l_1l_2 + m_1m_2|}{\sqrt{l_1^2l_2^2 + l_1^2m_2^2 + l_2^2m_1^2 + m_1^2m_2^2}}$$

$$= \frac{|l_1l_2 + m_1m_2|}{\sqrt{(l_1l_2 + m_1m_2)^2 + 2l_1l_2m_1m_2 + (l_1m_2 + l_2m_1)^2 - 2l_1m_2m_2l_1}}$$

$$= \frac{|a + b|}{\sqrt{(a + b)^2 + (2h)^2}} \quad \because l_1l_2 = a, \quad m_1m_2 = b, \quad l_1m_2 + l_2m_1 = 2h$$

$$= \frac{|a + b|}{\sqrt{(a + b)^2 + 4h^2}}$$

2. Prove that the product of perpendiculars from ( ) to the pair of straight lines

$$ax^2 + 2hxy + by^2 = 0 \text{ is}$$

$$\frac{|a^2 + 2h^2 + b^2|}{\sqrt{(a + b)^2 + 4h^2}}$$

Sol:  $ax^2 + 2hxy + by^2 = 0$  Let the pair of straight lines represents the lines

$$l_1x + m_1y = 0$$

$$l_2x + m_2y = 0$$

$$ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$$

$$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$$

by comparing like term coefficients on both sides

$$l_1l_2 = a, \quad l_1m_2 + l_2m_1 = 2h, \quad m_1m_2 = b$$

Let  $d_1$  = perpendicular distance from a point to  $l_1x + m_1y = 0$

$$d_1 = \frac{|l_1x + m_1y|}{\sqrt{l_1^2 + m_1^2}}$$

similarly

Let  $d_2$  = perpendicular distance from a point to  $l_2x + m_2y = 0$

$$d_2 = \frac{|l_2x + m_2y|}{\sqrt{l_2^2 + m_2^2}}$$

Product of perpendicular distances =  $d_1 \times d_2$

$$\begin{aligned} & \frac{|l_1x + m_1y|}{\sqrt{l_1^2 + m_1^2}} \times \frac{|l_2x + m_2y|}{\sqrt{l_2^2 + m_2^2}} \\ & \frac{|(l_1x + m_1y)(l_2x + m_2y)|}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}} \\ & \frac{|l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2|}{\sqrt{l_1^2l_2^2 + l_1^2m_2^2 + l_2^2m_1^2 + m_1^2m_2^2}} \\ & \frac{|l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2|}{\sqrt{(l_1l_2 + m_1m_2)^2 - 2l_1l_2m_1m_2 - (l_1m_2 + l_2m_1)^2 + 2l_1l_2m_1m_2}} \\ & \frac{|l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2|}{\sqrt{(l_1l_2 + m_1m_2)^2 - (l_1m_2 + l_2m_1)^2}} \\ & \frac{|a^2x^2 + 2hxy + b^2y^2|}{\sqrt{(a+b)^2 - (2h)^2}} \quad \because l_1l_2 = a, m_1m_2 = b, l_1m_2 + l_2m_1 = 2h \\ & \frac{|a^2x^2 + 2hxy + b^2y^2|}{\sqrt{(a+b)^2 - 4h^2}} \end{aligned}$$

3. Show that the area of a triangle formed by the lines  $ax^2 + 2hxy + by^2 = 0$  and

$$lx + my + n = 0 \text{ is } \left| \frac{n^2 \sqrt{h^2 - ab}}{am^2 - 2hlm - bl^2} \right|$$

**Sol:** Let the equation  $ax^2 + 2hxy + by^2 = 0$  represents the lines

$$l_1x + m_1y + n_1 = 0$$

$$l_2x + m_2y + n_2 = 0$$

$$ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$$

$$l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$$

Let the given line be  $lx + my + n = 0$

$$l_1l_2 = a, l_1m_2 + l_2m_1 = 2h, m_1m_2 = b$$

Let A be the intersection point of (1) and (3)

$$\frac{x}{m_1n} + \frac{y}{nl_1} = \frac{1}{l_1m + lm_1}$$

$$x = \frac{m_1n}{l_1m + lm_1}, y = \frac{nl_1}{l_1m + lm_1}$$

$$A = \left( \frac{m_1n}{l_1m + lm_1}, \frac{nl_1}{l_1m + lm_1} \right)$$

Similarly, Let B be the intersection point of (2) and (3)

$$\frac{x}{m_2n} + \frac{y}{nl_2} = \frac{1}{l_2m + lm_2}$$

$$B = \left( \frac{m_2n}{l_2m + lm_2}, \frac{nl_2}{l_2m + lm_2} \right)$$

clearly  $O(0,0)$  be the point of intersection of (1) and (2)

[Area of a triangle whose vertices  $(0, 0), (x_1, y_1), (x_2, y_2)$  is  $\frac{1}{2}|x_1y_2 - x_2y_1|$ ]

$$\text{Area of } \triangle OAB = \frac{1}{2} \left| \frac{m_1n}{l_1m + lm_1} \cdot \frac{nl_2}{l_2m + lm_2} - \frac{m_2n}{l_2m + lm_2} \cdot \frac{nl_1}{l_1m + lm_1} \right|$$

$$= \frac{1}{2} \left| \frac{l_1m_2n^2 - m_1l_2n^2}{(l_1m + lm_1)(l_2m + lm_2)} \right|$$

$$= \frac{1}{2} \left| \frac{n^2(l_1m_2 - m_1l_2)}{l_1l_2m^2 + (l_1m_2 + l_2m_1)lm + m_1m_2l^2} \right|$$

$$= \frac{1}{2} \left| \frac{n^2\sqrt{(l_1m_2 - l_2m_1)^2}}{l_1l_2m^2 + (l_1m_2 + l_2m_1)lm + m_1m_2l^2} \right|$$

$$= \frac{1}{2} \left| \frac{n^2\sqrt{(l_1m_2 - l_2m_1)^2 - 4l_1m_2l_2m_1}}{l_1l_2m^2 + (l_1m_2 + l_2m_1)lm + m_1m_2l^2} \right|$$

$$\frac{1}{2} \left| \frac{n^2 \sqrt{(2h)^2 - 4ab}}{am^2 - 2hlm - bl^2} \right| \quad \because l_1 l_2 = a, m_1 m_2 = b, l_1 m_2 + l_2 m_1 = 2h$$

$$\frac{1}{2} \left| \frac{n^2 \sqrt{4h^2 - 4ab}}{am^2 - 2hlm - bl^2} \right|$$

$$\left| \frac{n^2 \sqrt{h^2 - ab}}{am^2 - 2hlm - bl^2} \right|$$

4. Show that the straight lines represented by  $(x + 2a)^2 - 3y^2 = 0$  and  $x = 0$  form an equilateral triangle.

**Sol:** Given equation of pair of straight lines is  $(x + 2a)^2 - (\sqrt{3}y)^2 = 0$

$$(x + 2a + \sqrt{3}y)(x + 2a - \sqrt{3}y) = 0$$

The lines represented by the given equation of pair of straight lines are

$$x + \sqrt{3}y + 2a = 0$$

$$x - \sqrt{3}y + 2a = 0$$

Given line is  $x - a = 0$

Let A be the angle between (1) and (3) then

$$\cos A = \frac{|1 \cdot 1 + \sqrt{3} \cdot 0|}{\sqrt{1 + 3} \sqrt{1 + 0}} = \frac{1}{2} \quad A = 60^\circ$$

Again B be the angle between (2) and (3) then

$$\cos B = \frac{|1 \cdot 1 + (-\sqrt{3}) \cdot 0|}{\sqrt{1 + 3} \sqrt{1 + 0}} = \frac{1}{2} \quad B = 60^\circ$$

Since (1) and (2) are different lines and two angles are  $60^\circ, 60^\circ$

Third angle is also  $60^\circ$

Given lines form an equilateral triangle.

5. Find the centroid and area of a triangle formed by the lines  $2y^2 - xy - 6x^2 = 0$  and  $x + y + 4 = 0$

**Sol:** Given equation of straight lines  $2y^2 - xy - 6x^2 = 0$  .....(1)

$$2y^2 - 4xy + 3xy - 6x^2 = 0$$

$$2y(y - 2x) + 3x(y - 2x) = 0$$

$$(y - 2x)(2y + 3x) = 0$$

The lines represented by the given equation  $2y^2 - xy - 6x^2 = 0$  are given by

$$2x - y = 0 \quad \dots(2)$$

$$\text{and } 3x - 2y = 0 \quad \dots(3)$$

$$\text{and given straight line } x + y + 4 = 0 \quad \dots(4)$$

Clearly  $O(0, 0)$  is point of intersection of (2) and (3)

Let A be the point of intersection of (2) and (4)

$$\frac{x}{4} = \frac{y}{0} = \frac{1}{2} = \frac{1}{1}$$

$$A = \left( \frac{4}{3}, \frac{8}{3} \right)$$

B be the point of intersection of (3) and (4)

$$\frac{x}{8} = \frac{y}{0} = \frac{1}{3} = \frac{1}{2}$$

$$B = (8, 12)$$

$$\text{Centroid of } \triangle ABC = \left( \frac{0 + \frac{4}{3} + 8}{3}, \frac{0 + \frac{8}{3} + 12}{3} \right) = \left( \frac{20}{9}, \frac{44}{9} \right)$$

[Area of a triangle whose vertices are  $O(0, 0)$ ,  $A(x_1, y_1)$ ,  $B(x_2, y_2)$  is  $\frac{1}{2}|x_1y_2 - x_2y_1|$ ]

$$\begin{aligned} \text{Required area of triangle} &= \frac{1}{2} \left| \frac{4}{3} \cdot 12 - 8 \cdot \frac{8}{3} \right| \\ &= \frac{1}{2} \left| \frac{48}{3} - \frac{64}{3} \right| = \frac{1}{2} \left| \frac{112}{3} - \frac{56}{3} \right| \text{ Sq.units} \end{aligned}$$

6. Show that the lines represented by  $(lx + my)^2 - 3(mx - ly) = 0$  and  $lx + my = 0$  form

an equilateral triangle with area  $\frac{n^2}{\sqrt{3}(l^2 + m^2)}$

Sol: Given straight line  $lx + my = 0 \quad \dots(1)$

$$\text{Slope of } m_1 = \frac{l}{m}$$

Let the equation of line  $y = m_2x$  which makes an angle of  $60^\circ$  with (1)

$$\tan \left| \frac{m_1 - m_2}{1 + m_1m_2} \right|$$

$$\tan 60^\circ = \frac{\left| \frac{l/m}{1} \frac{y/x}{l/m \ y/x} \right|}{\left| \frac{l/m}{1} \frac{y/x}{l/m \ y/x} \right|} \quad \because m_2 = \frac{y}{x}$$

$$\sqrt{3} = \frac{\left| \frac{lx - my}{mx} \right|}{\left| \frac{lx - my}{mx} \right|} \left| \frac{lx - my}{mx} \right|$$

squaring both sides

$$3(mx - ly)^2 = (lx - my)^2$$

$$(lx - my)^2 - 3(mx - ly)^2 = 0$$

which is given equation of pair of straight in the data

$(lx - my)^2 - 3(mx - ly)^2 = 0$  and  $lx + my + n = 0$  form an equilateral triangle

Also O is point of intersection of lines in (2)

Perpendicular distance to (1) from O(0,0) P =  $\frac{|n|}{\sqrt{l^2 + m^2}}$

Area of that equilateral triangle

$$\frac{P^2}{\sqrt{3}} = \frac{n^2}{\sqrt{3}(l^2 + m^2)}$$

7. If ( ) is the centroid of the triangle formed by the lines  $ax^2 + 2hxy + by^2 = 0$  and

$lx + my = 0$  then prove that  $\frac{bl}{am} = \frac{hm}{hl} = \frac{2}{3(bl^2 + 2hlm + am^2)}$

**Sol:** Let  $ax^2 + 2hxy + by^2 = 0$  represent the lines

$$l_1x + m_1y = 0$$

$$l_2x + m_2y = 0$$

$$ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$$

$$l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$$

comparing both sides like terms

$$l_1l_2 = a, l_1m_2 + l_2m_1 = 2h, m_1m_2 = b$$

Given equation  $lx + my = 0$

Clearly O(0, 0) is point of intersection of (1) and (2)

Let A be the point of intersection of (1) and (3)



$$\frac{x}{m_1} = \frac{y}{0} = \frac{1}{ml_1 + lm_1}$$

$$A \left( \frac{m_1}{l_1 m + lm_1}, \frac{l_1}{l_1 m + lm_1} \right)$$

Again let B be the point of intersection of (2) and (3)

$$\frac{x}{m_2} = \frac{y}{0} = \frac{1}{ml_2 + lm_2}$$

$$B \left( \frac{m_2}{l_2 m + lm_2}, \frac{l_2}{l_2 m + lm_2} \right)$$

centroid of OAB =

$$\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

$$\left( \frac{1}{3} \left( \frac{m_1}{l_1 m + lm_1} + \frac{m_2}{l_2 m + lm_2} \right), \frac{1}{3} \left( \frac{l_1}{l_1 m + lm_1} + \frac{l_2}{l_2 m + lm_2} \right) \right)$$

$$\frac{1}{3} \left( \frac{m_1}{l_1 m + lm_1} + \frac{m_2}{l_2 m + lm_2} \right), \frac{1}{3} \left( \frac{l_1}{l_1 m + lm_1} + \frac{l_2}{l_2 m + lm_2} \right)$$

Now

$$\frac{1}{3} \frac{m_1(l_2 m + lm_2) + m_2(l_1 m + lm_1)}{(l_1 m + lm_1)(l_2 m + lm_2)}$$

$$\frac{1}{3} \frac{m_1 l_2 m + l m_1 m_2 + l_1 m m_2 + l m_1 m_2}{l_1 l_2 m^2 + (l_1 m_2 + l_2 m_1) l m + m_1 m_2 l^2}$$

$$\frac{1}{3} \frac{2 m_1 m l + m(l_1 m_2 + l_2 m_1)}{l_1 l_2 m^2 + (l_1 m_2 + l_2 m_1) l m + m_1 m_2 l^2}$$

$$\frac{1}{3} \frac{2 b l + m(2 h)}{a m^2 + 2 h l m + b l^2}$$

$$\frac{bl + hm}{3(bl^2 + 2hlm + am^2)}$$

similarly we can prove  $\frac{am + hl}{3(bl^2 + 2hlm + am^2)}$

$$\frac{bl + hm}{3(bl^2 + 2hlm + am^2)}$$

8. Prove that the equation  $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$  represents a pair of perpendicular lines.

**Sol:** compare the given equation  $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$  with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$a = 2, h = \frac{3}{2}, b = -2, g = \frac{3}{2}, f = \frac{1}{2}, c = 1$$

$$\text{Now } abc + 2fgh - af^2 - bg^2 - ch^2$$

$$2(-2)(1) + 2\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^2 - (-2)\left(\frac{3}{2}\right)^2 - 1\left(\frac{3}{2}\right)^2$$

$$= 4 - \frac{9}{2} - \frac{1}{2} + \frac{9}{2} - \frac{9}{4}$$

$$= 4 - \frac{8}{2} = 0$$

$$\text{also } h^2 - ab = \left(\frac{9}{4}\right) - 2(-2) = \frac{9}{4} + 4 = \frac{9 + 16}{4} = \frac{25}{4} > 0$$

$$g^2 - ac = \left(\frac{9}{4}\right) - 2(1) = \frac{9}{4} - 2 = \frac{9 - 8}{4} = \frac{1}{4} > 0$$

$$f^2 - bc = \left(\frac{1}{4}\right) - (-2) = \frac{1}{4} + 2 = \frac{1 + 8}{4} = \frac{9}{4} > 0$$

given equation  $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$  represents a pair of straight line also  
coeff.  $x^2$  + coeff.  $y^2$

The lines in the pair of straight lines are perpendicular.

9. Find the values of k, if the lines joining the origin to the points of intersection of the curve  $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$  and the line  $x + 2y = k$  are mutually perpendicular

**Sol:** Given straight line equation  $x + 2y = k$   $\frac{x + 2y}{k} = 1$  ..... (1)

Given curve equation  $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$  ..... (2)

To get the required equation by homogenising (2) with the help of equation (1)

$$2x^2 - 2xy + 3y^2 + (2x - y) \left(\frac{x + 2y}{k}\right) - 1\left(\frac{x + 2y}{k}\right)^2 = 0$$

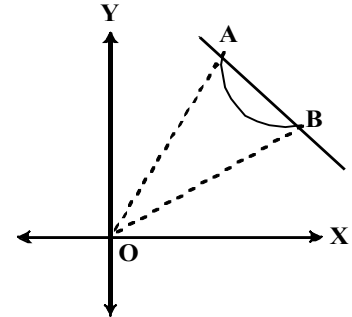
$$2x^2 - 2xy + 3y^2 + (2x - y) \frac{(x + 2y)}{k} - 1 \frac{(x + 2y)^2}{k^2} = 0$$

$$k^2(2x^2 - 2xy + 3y^2) - k(2x^2 - 4xy + xy + 2y^2) + (x^2 - 4y^2 + 4xy) = 0$$

$$x^2(2k^2 - 2k - 1) - xy(2k^2 - 3k - 4) + y^2(3k^2 - 2k - 4) = 0$$

If the two lines in above equation are perpendicular then

$$\begin{aligned} \text{coeff. } x^2 + \text{coeff. } y^2 &= 0 \\ 2k^2 + 2k - 1 + 3k^2 - 2k - 4 &= 0 \\ 5k^2 - 5 &= 0 \\ 5k^2 &= 5 \\ k^2 &= 1 \\ k &= \pm 1 \end{aligned}$$



10. Find the angle between the lines joining the origin to the points of intersection of the curve  $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$  and the line  $3x - y + 1 = 0$

Sol: Given straight line equation  $3x - y + 1 = 0 \implies y - 3x = 1 \dots\dots (1)$

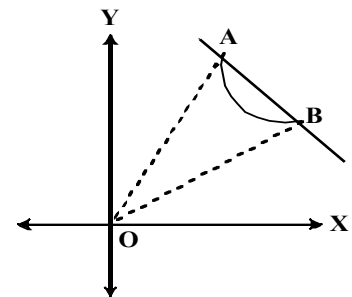
Given curve equation  $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0 \dots\dots (2)$

to get the equations of the lines OA, OB is obtained by homogenising the equation (2) by equation (1)

$$\begin{aligned} x^2 + 2xy + y^2 + 2x + 2y - 5(1^2) &= 0 \\ x^2 + 2xy + y^2 + (2x + 2y)(y - 3x) - 5(y - 3x)^2 &= 0 \\ x^2 + 2xy + y^2 + 2xy - 6x^2 + 2y^2 - 6xy - 5(y^2 - 9x^2 - 6xy) &= 0 \\ x^2 + 2xy + y^2 + 2xy - 6x^2 + 2y^2 - 6xy - 5y^2 + 45x^2 + 30xy &= 0 \\ 50x^2 + 28xy + 2y^2 &= 0 \\ 25x^2 + 14xy + y^2 &= 0 \end{aligned}$$

If  $\theta$  is the angle between the lines in the above equation then

$$\begin{aligned} \cos \theta &= \frac{|a - b|}{\sqrt{a^2 + b^2 + 4h^2}} \\ a = 25, h = -7, b = 1 & \\ \cos \theta &= \frac{|25 - 1|}{\sqrt{25^2 + 1^2 + 4(-7)^2}} \\ &= \frac{24}{\sqrt{276 + 196}} \end{aligned}$$



## Three Dimensional Coordinates

### Key concepts

If  $P(x_1, y_1, z_1)$ ,  $Q(x_2, y_2, z_2)$  are two points in the space then distance between P and Q

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The point dividing the segment  $\overline{AB}$  where  $A(x_1, y_1, z_1)$ ,  $B(x_2, y_2, z_2)$  in the ratio  $m:n$  is

$$(i) \quad \frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}$$

(ii) Mid point of  $\overline{AB}$

$$\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}$$

Centroid of a triangle with vertices  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ ,  $(x_3, y_3, z_3)$

$$\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}$$

Centroid of a tetrahedron whose vertices are

$(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ ,  $(x_3, y_3, z_3)$ ,  $(x_4, y_4, z_4)$

$$\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}$$

### Very Short Answer Questions (2 Marks)

1. Find  $x$  if the distance between  $(5, -1, 7)$  and  $(x, 5, 1)$  is 9 units.

**Sol:** Let the given points be  $A(5, -1, 7)$ ,  $B(x, 5, 1)$

from problem, we have  $AB = 9$

$$AB^2 = 81$$

$$(x-5)^2 + (5+1)^2 + (1-7)^2 = 81$$

$$(x-5)^2 + 36 + 36 = 81$$

$$(x-5)^2 = 9$$

$$(x-5) = \pm 3$$

$$x = 5 \pm 3$$

$$x = 8, 2$$

2. Show that the points (1, 2, 3), (2, 3, 1), (3, 1, 2) form an equilateral triangle.

Sol: Let given points be A B C

$$AB = \sqrt{(2-1)^2 + (3-2)^2 + (1-3)^2} = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$$

$$BC = \sqrt{(3-2)^2 + (1-3)^2 + (2-1)^2} = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$$CA = \sqrt{(1-3)^2 + (2-1)^2 + (3-2)^2} = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$$

$$AB = BC = CA$$

ABC is an equilateral triangle

3. find the centroid of the triangle whose vertices are (5, 4, 6), (1, -1, 3), and (4, 3, 2)

Sol: Let given points be A (x<sub>1</sub>, y<sub>1</sub>, z<sub>1</sub>) = (5, 4, 6)

$$B (x_2, y_2, z_2) = (1, -1, 3)$$

$$C (x_3, y_3, z_3) = (4, 3, 2)$$

$$\begin{aligned} \text{Centroid of } ABC &= \left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right) \\ &= \left( \frac{5 + 1 + 4}{3}, \frac{4 + (-1) + 3}{3}, \frac{6 + 3 + 2}{3} \right) \\ &= \left( \frac{10}{3}, 2, \frac{11}{3} \right) \end{aligned}$$

4. Find the centroid of the tetrahedron whose vertices are (2, 3, -4), (-3, 3, -2), (-1, 4, 2) and (3, 5, 1)

Sol: Let the given points of a tetrahedron be

$$A (x_1, y_1, z_1) = (2, 3, -4)$$

$$B (x_2, y_2, z_2) = (-3, 3, -2)$$

$$C (x_3, y_3, z_3) = (-1, 4, 2)$$

$$D (x_4, y_4, z_4) = (3, 5, 1)$$

Centroid of a tetrahedron

$$\begin{aligned} \text{Centroid} &= \left( \frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4} \right) \\ &= \left( \frac{2 + (-3) + (-1) + 3}{4}, \frac{3 + 3 + 4 + 5}{4}, \frac{-4 + (-2) + 2 + 1}{4} \right) \\ &= \left( \frac{1}{4}, \frac{15}{4}, \frac{-3}{4} \right) \end{aligned}$$

5. Find the fourth vertex of the parallelogram whose consecutive vertices are **(2, 4, -1), (3, 6, -1) and (4, 5, 1)**

**Sol:** Let ABCD be parallelogram where A(2, 4, -1), B(3, 6, -1), C(4, 5, 1) and D(a, b, c)  
mid point of AC = mid point of BD

$$\frac{2+4}{2}, \frac{4+5}{2}, \frac{-1+1}{2} = \frac{3+a}{2}, \frac{6+b}{2}, \frac{-1+c}{2}$$

$$\frac{6}{2}, \frac{9}{2}, 0 = \frac{a+3}{2}, \frac{b+6}{2}, \frac{c-1}{2}$$

$$\frac{a-3}{2}, \frac{b-6}{2}, \frac{9-c}{2} = 0$$

$$a=3, b=3, c=1 \quad \text{th vertex } D(a, b, c) = (3, 3, 1)$$

6. Find the coordinates of the vertex 'C' of ABC if its centroid is the origin and the vertices A, B are **(1, 1, 1) and (-2, 4, 1)** respectively.

**Sol:** Let given points be A(1, 1, 1), B(-2, 4, 1), C(x, y, z)

Given that centroid of ABC = (0, 0, 0)

$$\frac{1-2+x}{3}, \frac{1+4+y}{3}, \frac{1+1+z}{3} = (0,0,0)$$

$$\frac{x-1}{3}, \frac{y-5}{3}, \frac{z-2}{3} = (0,0,0)$$

$$\frac{a-3}{2}, \frac{b-6}{2}, \frac{9-c}{2} = 0$$

$$1-x=0, 5-y=0, z-2=0$$

$$x=1, y=5, z=2 \quad \text{third vertex } C(x, y, z) = (1, -5, -2)$$

7. If **(3, 2, -1), (4, 1, 1), (6, 2, 5)** are three vertices and **(4, 2, 2)** is the centroid of a tetrahedron, find the fourth vertex.

**Sol:** Let the given points be A(3, 2, -1), B(4, 1, 1), C(6, 2, 5), D(x, y, z)

Given that centroid of ABCD tetrahedron = (4, 2, 2)

$$\frac{3+4+6+x}{4}, \frac{2+1+2+y}{4}, \frac{-1+1+5+z}{4} = (4,2,2)$$

$$\frac{13-x}{4}, \frac{5-y}{4}, \frac{5-z}{4} = (4,2,2) \quad \frac{13-x}{4} = 4, \frac{5-y}{4} = 2, \frac{5-z}{4} = 2$$

$$x=3, y=3, z=3 \quad \text{4th vertex } D(x, y, z) = (3, 3, 3)$$

\*\*\*\*\*

## Direction Cosines and Direction Ratios

### Key concepts

A ray  $\overline{OP}$  passing through origin  $O$  and making angles  $\alpha, \beta, \gamma$  respectively with  $\overline{OX}, \overline{OY}, \overline{OZ}$  then the numbers  $\cos \alpha, \cos \beta, \cos \gamma$  are called the direction cosines of the ray  $\overline{OP}$ . Usually they are denoted by  $l, m, n$  where  $l = \cos \alpha, m = \cos \beta, n = \cos \gamma$ .

If  $l, m, n$  are direction cosines of  $\overline{OP}$ , then direction cosines of  $\overline{PO}$  are  $-l, -m, -n$ .

If  $l, m, n$  are direction cosines of a line then  $l^2 + m^2 + n^2 = 1$ .

Triad of numbers  $(a, b, c)$  proportional to direction cosines of a line are called its direction ratios.

Direction cosines of a line whose direction ratios  $a, b, c$  are

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Direction ratios of the line joining  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$

$(x_2 - x_1, y_2 - y_1, z_2 - z_1)$  and its direction cosines are

$$\frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}, \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}, \frac{z_2 - z_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

If  $\theta$  is angle between two lines whose direction cosines are  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$  then  $\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$ .

If  $\theta$  is angle between two lines whose direction ratios are  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  then

$$\cos \theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

### Long Answer Questions (7 Marks)

1. Find the direction cosines of two lines which are connected by the relations  $l + m + n = 0$  and  $mn - 2nl - 2lm = 0$

Sol: Given that  $l + m + n = 0$  ....(1)

$$mn - 2nl - 2lm = 0 \quad \dots(2)$$

$$l = -m - n$$

From (1)  $l = -m - n$  substituting  $l$  value in we get

$$mn - 2n(-m - n) - 2(-m - n)m = 0$$

$$mn + 2mn + 2n^2 + 2m^2 + 2mn = 0$$

$$2m^2 + 4mn + mn + 2n^2 = 0$$

$$2m(m + 2n) + n(m + 2n) = 0$$

$$(m + 2n)(2m + n) = 0$$

$$m + 2n = 0 \text{ or } 2m + n = 0$$

Now  $2m + n = 0$   $n = -2m$

from  $l = -m - (-2m) = m$

$$l : m : n = m : m : -2m = 1 : 1 : -2$$

directional cosines  $\frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, \frac{2}{\sqrt{1^2 + 1^2 + (-2)^2}}$

$$\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$$

Now  $m + 2n = 0$   $m = -2n$

from (1)  $l = m + n = -2n + n = -n$

$$l : m : n = -n : -2n : n = 1 : 2 : 1$$

Direction cosines of a line is  $\frac{1}{\sqrt{1^2 + (-2)^2 + 1^2}}, \frac{2}{\sqrt{1^2 + (-2)^2 + 1^2}}, \frac{1}{\sqrt{1^2 + (-2)^2 + 1^2}}$

$$\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$$

Thus the d.c's of the two lines are  $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$  and  $\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$

2. Find the direction cosines of two lines which are connected by the relations  $l - 5m + 3n = 0$  and  $7l^2 + 5m^2 - 3n^2 = 0$



**Sol:** Given that  $l - 5m + 3n = 0$  ....(1)

$$7l^2 + 5m^2 - 3n^2 = 0 \quad \dots(2)$$

From  $l = 5m - 3n$  substitute this value in

$$\begin{aligned} &7(5m - 3n)^2 - 5m^2 - 3n^2 = 0 \\ &7(25m^2 - 9n^2 - 30mn) - 5m^2 - 3n^2 = 0 \\ &175m^2 - 63n^2 - 210mn - 5m^2 - 3n^2 = 0 \\ &180m^2 - 210mn - 60n^2 = 0 \\ &6m^2 - 7mn - 2n^2 = 0 \\ &(3m - 2n)(2m - n) = 0 \\ &3m - 2n = 0 \quad 2m - n = 0 \end{aligned}$$

Now  $3m - 2n = 0 \quad m = \frac{2}{3}n$

From  $l = 5m - 3n = \frac{10}{3}n - 3n = \frac{10n - 9n}{3} = \frac{n}{3}$

$$l : m : n = \frac{n}{3} : \frac{2}{3}n : n = \frac{1}{3} : \frac{2}{3} : 1 = 1 : 2 : 3$$

Direction cosines  $\frac{1}{\sqrt{1^2 + 2^2 + 3^2}}, \frac{2}{\sqrt{1^2 + 2^2 + 3^2}}, \frac{3}{\sqrt{1^2 + 2^2 + 3^2}}$

$$\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$$

similarly  $2m - n = 0 \quad n = 2m$

$$l = 5m - 3(2m) = 5m - 6m = -m$$

$$l : m : n = -m : m : 2m = -1 : 1 : 2$$

Direction cosines of another line  $\frac{1}{\sqrt{(-1)^2 + 1^2 + 2^2}}, \frac{1}{\sqrt{(-1)^2 + 1^2 + 2^2}}, \frac{2}{\sqrt{(-1)^2 + 1^2 + 2^2}}$

$$\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$$

**3. Find the angle between the lines whose direction cosines satisfy the equations**

$$l + m + n = 0, \quad l^2 + m^2 - n^2 = 0$$

**Sol:** Given equations  $l + m + n = 0$  ....(1)

$$l^2 + m^2 - n^2 = 0 \quad \dots(2)$$

From  $l = -m - n$  substitute this value in

$$\begin{aligned} (m+n)^2 - m^2 - n^2 &= 0 \\ m^2 + n^2 + 2mn - m^2 - n^2 &= 0 \\ 2m^2 + 2mn &= 0 \\ 2m(m+n) &= 0 \\ m=0 \text{ or } m+n &= 0 \end{aligned}$$

Now if  $m = 0$  From  $l = 0 - n = -n$

$$l : m : n = -n : 0 : n = -1 : 0 : 1$$

direction ratios of a line  $(a_1, b_1, c_1) = (-1, 0, 1)$

$$\begin{aligned} m : n &= 0 : m \\ 1 : m : n &= (n) : n : n \\ 1 : m : n &= 0 : n : n \end{aligned}$$

direction ratios of another line  $(a_2, b_2, c_2) = (0, 1, 1)$

Let  $\theta$  be the angle between two lines then

$$\begin{aligned} \cos \theta &= \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{|0(-1) + 0(0) + 1(1)|}{\sqrt{1+0+1} \sqrt{0+1+1}} \\ &= \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \\ \cos^{-1} \frac{1}{2} &= 60^\circ \end{aligned}$$

4. Find the angle between the lines whose direction cosines are given by the equations  $3l + m + 5n = 0$  and  $6mn - 2nl + 5lm = 0$

**Sol:** Given equation  $3l + m + 5n = 0$  ....(1)  
 $6mn - 2nl + 5lm = 0$  ....(2)

From  $m = -3l - 5n$  substitute this value in

$$\begin{aligned} 6(-3l-5n)n - 2nl + 5l(-3l-5n) &= 0 \\ 18ln - 30n^2 - 2nl - 15l^2 - 25ln &= 0 \\ 15l^2 - 45ln - 30n^2 &= 0 \\ l^2 - 3ln - 2n^2 &= 0 \\ (l-2n)(l+n) &= 0 \\ l-2n=0 \quad l+n &= 0 \end{aligned}$$

Now  $l + 2n = 0$   $l = -2n$

From  $m = -3l - 5n = 6n - 5n = n$

$$l : m : n = -2n : n : n = -2 : 1 : 1$$

Direction ratios of a line  $(a_1, b_1, c_1) = (-2, 1, 1)$

also  $l + n = 0 \quad l = -n$

from  $m = -3l - 5n = -3(-n) - 5n = 3n - 5n = -2n$

$$l : m : n = -n : -2n : n = -1 : -2 : 1 = 1 : 2 : -1$$

direction ratios of another line  $(a_2, b_2, c_2) = (1, 2, -1)$

If  $\theta$  is angle between two lines then

$$\cos \theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{|(-2)(1) + 1(2) + 1(-1)|}{\sqrt{(-2)^2 + 1^2 + 1^2} \sqrt{1^2 + 2^2 + (-1)^2}}$$

$$= \frac{|-2 + 2 - 1|}{\sqrt{6} \sqrt{6}} = \frac{-1}{6}$$

$$\cos^{-1} \left( \frac{1}{6} \right)$$

**5. Find the angle between two diagonals of a cube**

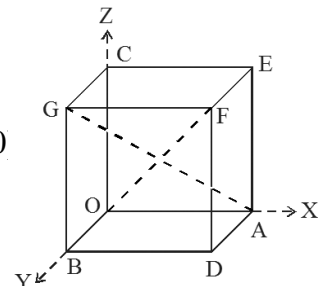
**Sol:** Let 'O', One of the vertices of the cube, be taken as the origin and the three coterminous edges  $\overline{OA}, \overline{OB}, \overline{OC}$  as coordinate axes.

let  $\overline{OA} = \overline{OB} = \overline{OC} = a$

Coordinates of vertices of cube  $O(0, 0, 0), A(a, 0, 0), B(0, a, 0)$

$C(0, 0, a), D(a, a, 0), E(a, 0, a), F(a, a, a), G(0, a, a)$

Also four diagonals of cube are  $\overline{OF}, \overline{AG}, \overline{BE}, \overline{CD}$



Now direction ratio's of diagonal  $\overline{OF}$  are  $(a - 0, a - 0, a - 0) = (a, a, a)$

direction ratio's of diagonal  $\overline{AG}$   $(0 - a, a - 0, a - 0) = (-a, a, a)$

If  $\theta$  is the angle between  $\overline{OF}, \overline{AG}$  then by

$$\cos \theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{|a(-a) + a(a) + a(a)|}{\sqrt{a^2 + a^2 + a^2} \sqrt{a^2 + a^2 + a^2}}$$

$$= \frac{a^2}{\sqrt{3}a \cdot \sqrt{3}a} = \frac{1}{3}$$

$$\cos^{-1} \left( \frac{1}{3} \right)$$

6. If a ray makes angles with the four diagonals of a cube

find  $\cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_3 + \cos^2 \theta_4$ .

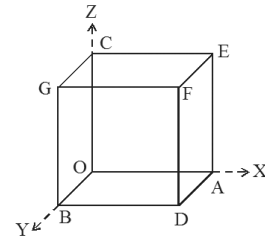
**Sol:** Let 'O', One of the vertices of the cube, be taken as the origin and the three coterminal edges  $\overline{OA}$ ,  $\overline{OB}$ ,  $\overline{OC}$  as coordinate axes.

let  $\overline{OA} = \overline{OB} = \overline{OC} = a$

Coordinates of vertices of cube  $O(0, 0, 0)$ ,  $A(a, 0, 0)$ ,  $B(0, a, 0)$ ,

$C(0, 0, a)$ ,  $D(a, a, 0)$ ,  $E(a, 0, a)$ ,  $F(a, a, a)$ ,  $G(0, a, a)$

Also four diagonals of cube are  $\overline{OF}$ ,  $\overline{AG}$ ,  $\overline{BE}$ ,  $\overline{CD}$



Now direction ratio's of diagonal  $\overline{OF}$  are  $(a - 0, a - 0, a - 0) = (a, a, a)$

direction ratio's of diagonal  $\overline{AG}$   $(0 - a, a - 0, a - 0) = (-a, a, a)$

direction ratio's of diagonal  $\overline{BE}$   $(a - 0, a - 0, a - 0) = (a, -a, a)$

direction ratio's of diagonal  $\overline{CD}$   $(a - 0, a - 0, 0 - a) = (a, a, -a)$

If any ray having direction ratio's  $(l, m, n)$  and makes an angle with four diagonals  $\overline{OF}$ ,  $\overline{AG}$ ,  $\overline{BE}$ ,  $\overline{CD}$  are then

$$\cos \theta = \frac{|al \quad am \quad an|}{\sqrt{a^2 \quad a^2 \quad a^2} \sqrt{l^2 \quad m^2 \quad n^2}} = \frac{|a(l \quad m \quad n)|}{\sqrt{3a^2} \sqrt{l^2 \quad m^2 \quad n^2}}$$

$$\cos^2 \theta = \frac{a^2(l \quad m \quad n)^2}{3a^2(l^2 \quad m^2 \quad n^2)} = \frac{(l \quad m \quad n)^2}{3(l^2 \quad m^2 \quad n^2)}$$

similarly

$$\cos^2 \theta_1 = \frac{(l \quad m \quad n)^2}{3(l^2 \quad m^2 \quad n^2)}, \cos^2 \theta_2 = \frac{(l \quad m \quad n)^2}{3(l^2 \quad m^2 \quad n^2)}, \cos^2 \theta_3 = \frac{(l \quad m \quad n)^2}{3(l^2 \quad m^2 \quad n^2)}$$

$$\cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_3 + \cos^2 \theta_4 = \frac{(l \quad m \quad n)^2}{3(l^2 \quad m^2 \quad n^2)} + \frac{(l \quad m \quad n)^2}{3(l^2 \quad m^2 \quad n^2)} + \frac{(l \quad m \quad n)^2}{3(l^2 \quad m^2 \quad n^2)} + \frac{(l \quad m \quad n)^2}{3(l^2 \quad m^2 \quad n^2)}$$

simplification

$$\frac{4l^2 \quad 4m^2 \quad 4n^2}{3(l^2 \quad m^2 \quad n^2)} = \frac{4(l^2 \quad m^2 \quad n^2)}{3(l^2 \quad m^2 \quad n^2)} = \frac{4}{3}$$

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## The Plane

### Key concepts

A plane is a surface with at least three non collinear points such that the line joining any two points on the surface lies entirely on it.

Equation of the plane in normal form is  $lx + my + nz = p$  where  $l, m, n$  are direction cosines of the normal to the plane and 'p' is the perpendicular distance to the plane from the origin.

General equation of the plane is  $ax + by + cz + d = 0$  where  $a, b, c$  are direction ratios of the normal to the plane.

Equation of the plane in intercept form  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$  where  $a, b, c$  are intercepts on X, Y,

Z – axes respectively.

Angle between two planes is the angle between their normals. If  $\theta$  is the angle between the planes  $a_1x + b_1y + c_1z + d_1 = 0$  and  $a_2x + b_2y + c_2z + d_2 = 0$  then

$$\cos \theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(i) If  $a_1a_2 + b_1b_2 + c_1c_2 = 0$  then planes are perpendicular.

(ii) If  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$  then the planes are parallel.

### Very Short Answer Questions (2 Marks)

1. Reduce the equation  $x + 2y - 3z - 6 = 0$  of the plane to the normal form.

**Sol:** Given equation of the plane  $x + 2y - 3z - 6 = 0$

dividing on both sides by  $\sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$

$$\frac{x}{\sqrt{14}} + \frac{2y}{\sqrt{14}} - \frac{3z}{\sqrt{14}} - \frac{6}{\sqrt{14}} = 0$$

$$\frac{1}{\sqrt{14}}x + \frac{2}{\sqrt{14}}y - \frac{3}{\sqrt{14}}z - \frac{6}{\sqrt{14}} = 0$$

2. Find the equation of the plane whose intercepts on X, Y, Z – axes are 1, 2, 4 respectively.

**Sol:** Let the intercepts on X, Y, Z – axes

are  $a = 1, b = 2, c = 4$  respectively then equation of the plane in intercept form is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{4} = 1$$

$$4x + 2y + z = 4$$

3. Write the equation of the plane  $4x - 4y + 2z + 5 = 0$  in the intercept form.

**Sol:** Given equation of the plane  $4x - 4y + 2z + 5 = 0$

$$4x - 4y + 2z = -5$$

$$\frac{4x}{5} - \frac{4y}{5} + \frac{2z}{5} = -1$$

$$\frac{4}{5}x - \frac{4}{5}y + \frac{2}{5}z = -1$$

$$\frac{x}{5/4} - \frac{y}{5/4} + \frac{z}{5/2} = -1$$

4. Find the angle between the planes  $x + 2y + 2z - 5 = 0$  and  $3x + 3y + 2z - 8 = 0$

**Sol:** Given equation of the plane  $x + 2y + 2z - 5 = 0$  .....(1)

$$a_1 = 1, b_1 = 2, c_1 = 2$$

and given another plane equation  $3x + 3y + 2z - 8 = 0$  .....(2)

$$a_2 = 3, b_2 = 3, c_2 = 2$$

Let  $\theta$  be the angle between the planes (1) and (2)

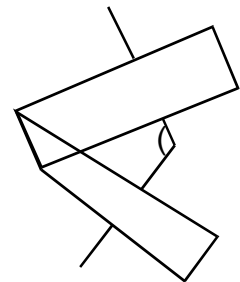
$$\text{then } \cos \theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\frac{|1 \cdot 3 + 2 \cdot 3 + 2 \cdot 2|}{\sqrt{1^2 + 2^2 + 2^2} \sqrt{3^2 + 3^2 + 2^2}}$$

$$\frac{|3 + 6 + 4|}{\sqrt{1 + 4 + 4} \sqrt{9 + 9 + 4}} = \frac{13}{3\sqrt{22}}$$

$$\cos^{-1} \frac{13}{3\sqrt{22}}$$

\*\*\*\*\*



## Limits

### Key concepts

$$* \quad \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$* \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (\text{x is in radians})$$

$$* \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad (\text{x is in radians})$$

$$* \quad \lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

$$* \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$* \quad \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$$

$$* \quad \lim_{x \rightarrow 0} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m} a^{n-m}$$

$$* \quad \lim_{x \rightarrow 0} \frac{\sin ax}{x} = a \quad (\text{x is in radians})$$

$$* \quad \lim_{x \rightarrow 0} \frac{\tan ax}{x} = a \quad (\text{x is in radians})$$

$$* \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

### Very Short Answer Questions (2 Marks)

$$\lim_{x \rightarrow 0} \frac{e^{x^3} - e^3}{x}$$

A.  $\lim_{x \rightarrow 0} \frac{e^x \cdot e^3 - e^3}{x} = \frac{a^m \cdot n}{a^m \cdot a^n}$

$$\lim_{x \rightarrow 0} e^3 \frac{(e^x - 1)}{x}$$

$$e^3 \lim_{x \rightarrow 0} \frac{e^x - 1}{x}$$

$$e^3 \cdot 1 = e^3$$

$$\lim_{x \rightarrow 0} \frac{e^x - \sin x - 1}{x}$$

A.  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \frac{\sin x}{x}$

$$= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$= 1 - 1 = 0$$

$$\lim_{x \rightarrow 0} \frac{\sin ax}{x \cos x}$$

$$\text{A. } \lim_{x \rightarrow 0} \frac{\sin ax}{x \cos x} = \frac{\lim_{x \rightarrow 0} \frac{\sin ax}{ax} \cdot a}{\lim_{x \rightarrow 0} \frac{x}{\cos x}} = \frac{a}{1} = a$$

$$\lim_{x \rightarrow 0} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 + 7}$$

$$\text{A. } \lim_{x \rightarrow 0} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 + 7}$$

$$\lim_{x \rightarrow 0} \frac{x^3 - 11 \left( \frac{3}{x^2} - \frac{4}{x^3} \right)}{x^3 - 13 \left( \frac{5}{x} - \frac{7}{x^3} \right)}$$

$$\text{As } x \rightarrow 0, \frac{1}{x}, \frac{1}{x^2} \text{ and } \frac{1}{x^3} \rightarrow \infty$$

$$\lim_{x \rightarrow 0} \frac{11 \left( \frac{3}{x^2} - \frac{4}{x^3} \right)}{13 \left( \frac{5}{x} - \frac{7}{x^3} \right)} = \frac{11 \cdot 0 - 0}{13 \cdot 0 - 0} = \frac{0}{0}$$

$$\lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x^2-1)}$$

$$\text{A. } \lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x^2-1)}$$

$$\lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x-1)} \cdot \lim_{x \rightarrow 1} \frac{1}{x+1}$$

Put  $y = x - 1$  so that as  $x \rightarrow 1, y \rightarrow 0$

$$\lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x-1)} = \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

$$1 \cdot \frac{1}{1+1} = \frac{1}{2}$$



$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} \quad b \neq 0, a \neq b$$

A. 
$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \lim_{x \rightarrow 0} \frac{\frac{\sin ax}{ax} \cdot ax}{\frac{\sin bx}{bx} \cdot bx}$$

$$= \frac{\lim_{x \rightarrow 0} \frac{\sin ax}{ax} \cdot a}{\lim_{x \rightarrow 0} \frac{\sin bx}{bx} \cdot b}$$

$$= \frac{1 \cdot a}{1 \cdot b} = \frac{a}{b}$$

$$\lim_{x \rightarrow 0} \frac{8|x| + 3x}{3|x| + 2x}$$

A. 
$$\lim_{x \rightarrow 0} \frac{8|x| + 3x}{3|x| + 2x} = \lim_{x \rightarrow 0} \frac{8x + 3x}{3x + 2x}$$

$$= \lim_{x \rightarrow 0} \frac{11x}{5x} = 11$$

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{b^x - 1}$$

A. 
$$\lim_{x \rightarrow 0} \frac{a^x - 1}{b^x - 1} = \lim_{x \rightarrow 0} \frac{\frac{a^x - 1}{x}}{\frac{b^x - 1}{x}}$$

$$= \frac{\lim_{x \rightarrow 0} \frac{a^x - 1}{x}}{\lim_{x \rightarrow 0} \frac{b^x - 1}{x}}$$

$$\frac{\log_e a}{\log_e b}$$

$$\log_b a$$

$$\lim_{x \rightarrow 0} \frac{e^{7x} - 1}{x}$$

A. As  $x \rightarrow 0$

$$7x \rightarrow 0$$

$$\lim_{7x \rightarrow 0} \frac{e^{7x} - 1}{7x} = 7 \cdot 7 \therefore \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1}{x}$$

A. For  $0 < |x| < 1$ , we have

$$\frac{\sqrt{x-1} - 1}{x} = \frac{\sqrt{1-x} - 1}{x} \cdot \frac{\sqrt{1-x} + 1}{\sqrt{1-x} + 1}$$

$$\frac{1-x-1}{x\sqrt{1-x}+1} = \frac{-x}{x\sqrt{1-x}+1}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{1-x} + 1}$$

$$\frac{1}{1+1} = \frac{1}{2}$$

$$\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x}$$

A. 
$$\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x} = \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{\sin x} \cdot \frac{\sin x}{x}$$

$$= \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{\sin x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$= \lim_{\sin x \rightarrow 0} \frac{e^{\sin x} - 1}{\sin x} \cdot 1$$

$$= 1 \cdot 1 = 1$$

$$\lim_{x \rightarrow 0} \frac{\log(1-5x)}{x}$$

A. 
$$\lim_{x \rightarrow 0} \frac{\log(1-5x)}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \cdot \log(1-5x) = \lim_{x \rightarrow 0} \log(1-5x)^{\frac{1}{x}} \quad \boxed{m \log x = \log x^m}$$

$$= \log \lim_{5x \rightarrow 0} (1-5x)^{\frac{1}{5x} \cdot 5}$$

$$= \log_e e^5$$

$$5 \log_e e$$

$$5 \quad \because \log_e e = 1$$

Show that  $\lim_{x \rightarrow 0} \frac{2|x|}{x} = 1$

A.  $\lim_{x \rightarrow 0} \frac{|x|}{x}$

$$= \lim_{x \rightarrow 0} \frac{2|x|}{x}$$

$$= \lim_{x \rightarrow 0} \frac{2x}{x} = 2$$

$$= \lim_{x \rightarrow 0} \frac{2x}{x} = 2$$

$$= 2$$

**Short Answer Questions (4 Marks)**

$$\lim_{x \rightarrow 3} \frac{x^2 - 8x + 15}{x^2 - 9}$$

A.  $\lim_{x \rightarrow 3} \frac{x^2 - 5x + 3x - 15}{(x - 3)(x + 3)}$

$$= \lim_{x \rightarrow 3} \frac{(x - 5)(x - 3)}{(x - 3)(x + 3)}$$

$$\frac{1}{3}$$

$$= \frac{1}{3}$$

Compute  $\lim_{x \rightarrow 0} \frac{(1 - x)^{\frac{1}{3}} - (1 + x)^{\frac{1}{3}}}{x}$

A.  $\lim_{x \rightarrow 0} \frac{(1 - x)^{\frac{1}{3}} - (1 + x)^{\frac{1}{3}}}{x}$

$$\lim_{x \rightarrow 0} \frac{(1 - x)^{\frac{1}{3}} - 1 - ((1 + x)^{\frac{1}{3}} - 1)}{x}$$

$$\lim_{x \rightarrow 0} \frac{(1 - x)^{\frac{1}{3}} - 1 - (1 + x)^{\frac{1}{3}} + 1}{x}$$

$$\lim_{x \rightarrow 1} \frac{(1-x)^{\frac{1}{3}} - 1}{(1-x) - 1} \quad \lim_{x \rightarrow 1} \frac{(1-x)^{\frac{1}{3}} - 1^{\frac{1}{3}}}{(1-x) - 1}$$

$$\frac{\frac{1}{3}1^{-2/3}}{3} = \frac{\frac{1}{3}1^{-2/3}}{3} = \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{3}$$

$$\lim_{x \rightarrow a} \frac{\sqrt{a-2x} - \sqrt{3x}}{a\sqrt{3a-x} - 2\sqrt{x}}$$

$$\text{A.} \quad \lim_{x \rightarrow a} \frac{\sqrt{a-2x} - \sqrt{3x}}{a\sqrt{3a-x} - 2\sqrt{x}} = \lim_{x \rightarrow a} \frac{\sqrt{a-2x} - \sqrt{3x}}{a\sqrt{3a-x} - 2\sqrt{x}} \cdot \frac{\sqrt{a-2x} + \sqrt{3x}}{\sqrt{a-2x} + \sqrt{3x}} = \lim_{x \rightarrow a} \frac{\sqrt{3a-x} - 2\sqrt{x}}{\sqrt{3a-x} + \sqrt{3x}}$$

$$= \lim_{x \rightarrow a} \frac{(a-2x-3x)}{a-3a-x-4x} \cdot \frac{\sqrt{a-2x} - \sqrt{3x}}{\sqrt{3a-x} + \sqrt{3x}}$$

$$= \lim_{x \rightarrow a} \frac{a-x}{a-3(a-x)} \cdot \lim_{x \rightarrow a} \frac{\sqrt{a-2x} - \sqrt{3x}}{a\sqrt{3a-x} - 2\sqrt{x}}$$

$$= \frac{1}{3} \cdot \frac{\cancel{\sqrt{3a}}}{\cancel{\sqrt{a}}} = \frac{1}{2\sqrt{3}}$$

$$\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2}$$

$$\text{A.} \quad \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2} = \lim_{x \rightarrow 0} \frac{2\sin \frac{ax-bx}{2} \sin \frac{ax+bx}{2}}{x^2}$$

$\cos C - \cos D$	$= 2\sin \frac{C-D}{2} \sin \frac{C+D}{2}$
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$$= 2 \lim_{x \rightarrow 0} \frac{\sin \frac{(a-b)x}{2}}{\frac{(a-b)x}{2}} \cdot \frac{\sin \frac{(a+b)x}{2}}{\frac{(a+b)x}{2}}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin \frac{(a-b)x}{2}}{\frac{(a-b)x}{2}} \cdot \frac{a-b}{2} \lim_{x \rightarrow 0} \frac{\sin \frac{(a+b)x}{2}}{\frac{(a+b)x}{2}} \cdot \frac{a+b}{2}$$

$$= 2 \frac{a-b}{2} \lim_{x \rightarrow 0} \frac{\sin \frac{a-b}{2} x}{\frac{a-b}{2} x} \cdot \frac{a+b}{2} \lim_{x \rightarrow 0} \frac{\sin \frac{a+b}{2} x}{\frac{a+b}{2} x} \cdot \frac{a+b}{2}$$

$$= 2 \cdot 1 \cdot \frac{a-b}{2} \cdot 1 \cdot \frac{a+b}{2}$$

$$= \frac{(a-b)(a+b)}{2}$$

$$\frac{b^2 - a^2}{2}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos 2mx}{\sin^2 nx} \quad (m, n \neq 0)$$

A.  $\lim_{x \rightarrow 0} \frac{1 - \cos 2mx}{\sin^2 nx} = \lim_{x \rightarrow 0} \frac{2 \sin^2 mx}{\sin^2 nx}$

$$2 \frac{\lim_{x \rightarrow 0} \frac{\sin mx}{mx}^2}{\lim_{x \rightarrow 0} \frac{\sin nx}{nx}^2} = \frac{2m^2}{n^2}$$

$$\lim_{x \rightarrow 0} \frac{\sin(a-bx) - \sin(a+bx)}{x}$$

A. Since  $\boxed{\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}}$ ,  $\lim_{x \rightarrow 0} \frac{\sin(a-bx) - \sin(a+bx)}{x}$

$$= \lim_{x \rightarrow 0} \frac{2 \cos \frac{a-bx+a+bx}{2} \sin \frac{(a-bx)-(a+bx)}{2}}{x}$$

$$= 2 \cos \frac{2a}{2} \lim_{x \rightarrow 0} \frac{\sin bx}{bx} \cdot b$$

$$= 2 \cos a \cdot b = 2b \cos a$$

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## Differentiation

### Key concepts

- |  |  |
|--|--|
| $* \frac{d}{dx}(uv) = u \frac{du}{dx} + v \frac{dv}{dx}$                     | $* \frac{d}{dx} \frac{u}{v} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ |
| $* \frac{d}{dx}(x^n) = nx^{n-1}$   | $* \frac{d}{dx}(x) = 1$  |
| $* \frac{d}{dx}(\sin x) = \cos x$  | $* \frac{d}{dx}(\cos x) = -\sin x$   |
| $* \frac{d}{dx}(\tan x) = \sec^2 x$  | $* \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$                         |
| $* \frac{d}{dx}(\sec x) = \sec x \tan x$                                     | $* \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$    |
| $* \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$                              | $* \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$                             |
| $* \frac{d}{dx}(\sec^{-1} x) = \frac{1}{ x \sqrt{x^2-1}}$                    | $* \frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{ x \sqrt{x^2-1}}$   |
| $* \frac{d}{dx}(e^x) = e^x$  | $* \frac{d}{dx}(a^x) = a^x \log a$   |
| $* \frac{d}{dx}(\log x) = \frac{1}{x}$                                       | $* \frac{d}{dx}(\sinh x) = \cosh x$  |
| $* \frac{d}{dx}(\cosh x) = \sinh x$  | $* \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$                          |
| $* \frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$                       | $* \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$     |
| $* \frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$ | $* \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1-x^2}}$                      |
| $* \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$                      | $* \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$                             |
| $* \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$                             | $* \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{ x \sqrt{1-x^2}}$    |
| $* \frac{d}{dx}(\operatorname{cosech}^{-1} x) = -\frac{1}{ x \sqrt{x^2-1}}$  |  |

Very Short Answer Questions (2 Marks)

1.  $y = \log(\sin(\log x))$ , then find  $\frac{dy}{dx}$

A.  $v = \log x, u = \sin v, y = \log u$

$$\frac{dy}{du} = \frac{1}{u}; \frac{du}{dv} = \cos v; \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \frac{1}{\sin(\log x)} \cdot \cos(\log x) \cdot \frac{1}{x} = \frac{\cot(\log x)}{x}$$

2.  $f(x) = \log(\sec x + \tan x)$ , then find  $f'(x)$

A.  $u = \sec x + \tan x$  and  $y = \log u$

$$\frac{dy}{du} = \frac{1}{u}; \frac{du}{dx} = \sec x \cdot \tan x + \sec^2 x$$

$$= \sec x(\sec x + \tan x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sec x + \tan x} \cdot \sec x(\sec x + \tan x) = \sec x$$

3. Find the derivatives of the following functions.

(i)  $\frac{d}{dx} \log(\tan 5x)$       (ii)  $\frac{d}{dx} \cos[\log x + e^x]$       (iii)  $x e^{\sinh y}$       (iv)  $\frac{d}{dx} \sin(\cos(x^2))$

A. (i)  $\frac{d}{dx} \log(\tan 5x) = \frac{1}{\tan 5x} \cdot \frac{d}{dx} \tan 5x$

$$= \frac{1}{\tan 5x} \sec^2 5x \cdot \frac{d}{dx} 5x$$

$$= 5 \frac{\sec^2 5x}{\tan 5x}$$

(ii)  $\frac{d}{dx} \cos[\log x + e^x]$

$$= \sin[\log x + e^x] \cdot \frac{d}{dx} [\log x + e^x]$$

$$= \sin[\log x + e^x] \cdot \left( \frac{1}{x} + e^x \right)$$

$$(iii) \quad x = e^{\sinh y}$$

$$\frac{dx}{dy} = \frac{d}{dy} e^{\sinh y}$$

$$\frac{dx}{dy} = e^{\sinh y} \frac{d}{dy} \cosh y$$

$$\frac{dx}{dy} = e^{\sinh y} \cosh y$$

$$\frac{dy}{dx} = \frac{1}{e^{\sinh y} \cosh y}$$

$$(iv) \quad \frac{d}{dx} \sin(\cos(x^2))$$

$$\cos(\cos(x^2)) \frac{d}{dx} \cos x^2$$

$$\cos(\cos(x^2)) (\sin x^2) \frac{d}{dx} x^2$$

$$\cos(\cos(x^2)) \sin(x^2) 2x$$

4.  $f(x) = 1 + x + x^2 + \dots + x^{100}$ , then find  $f'(1)$

A. Differentiate both sides with respect to 'x'

$$f'(x) = 1 + 2x + 3x^2 + \dots + 100x^{99}$$

$$f'(1) = 1 + 2 + 3 + \dots + 100$$

$$\frac{100 \cdot 101}{2} = 5050 \quad x = \frac{x(x-1)}{2}$$

5.  $f(x) = xe^x \sin x$  then find  $f'(x)$

A. Differentiate both sides with respect to 'x'

$$\frac{d}{dx} f(x) = \frac{d}{dx} xe^x \sin x$$

$$f'(x) = xe^x \frac{d}{dx} \sin x + x \sin x \frac{d}{dx} e^x + \sin x \frac{dx}{dx} \cdot e^x$$

$$xe^x \cos x + x \sin x e^x + e^x \sin x$$

6.  $f(x) = e^x$ ,  $g(x) = \sqrt{x}$ , Find the derivatives of the  $f(x)$  with respect to  $g(x)$

A.  $f(x) = e^x$ ,  $g(x) = 5x$

$$\frac{d}{dx} f(x) = \frac{d}{dx} e^x$$

$$\frac{d}{dx} g(x) = \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$



$$\frac{df(x)}{dg(x)} = \frac{\frac{df(x)}{dx}}{\frac{dg(x)}{dx}} = \frac{e^x}{\frac{1}{2\sqrt{x}}} = 2\sqrt{x}e^x$$

7.  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$  then show that  $\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$

A. Differentiate both sides with respect to 'x'

$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1} \frac{dy}{dx} = 0$$

$$x^{-\frac{1}{3}} + y^{-\frac{1}{3}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x^{\frac{1}{3}}}{y^{\frac{1}{3}}} = -\sqrt[3]{\frac{y}{x}}$$

8.  $y = \log(\cosh 2x)$  then find  $\frac{dy}{dx}$

A. Differentiate both sides with respect to 'x'

$$\frac{d}{dx}y = \frac{d}{dx}\log(\cosh 2x)$$

$$= \frac{1}{\cosh 2x} \frac{d}{dx}\cosh 2x$$

$$= \frac{\sinh 2x}{\cosh 2x} \frac{d}{dx}2x$$

$$= 2 \tanh 2x$$

9.  $x = a\cos^3 t$ ,  $y = a\sin^3 t$  then find  $\frac{dy}{dx}$

A. Differentiate both sides with respect to 'x'

$$\frac{dx}{dt} = 3a\cos^2 t \sin t; \quad \frac{dy}{dt} = 3\sin^2 t \cos t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \frac{dt}{dx}$$

$$= \frac{3\sin^2 t \cos t}{3a\cos^2 t \sin t}$$

$$= \tan t$$

## Short Answer Questions (4 Marks)

1. Find the derivatives of the following functions from the first principles.

A. (i)  $f(x) = \sin 2x$

$$f(x+h) = \sin 2(x+h) = \sin(2x+2h)$$

First principle

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin(2x+2h) - \sin 2x}{h}$$

$$\boxed{\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}}$$

$$\lim_{h \rightarrow 0} 2 \frac{\cos \frac{2x+2h+2h}{2} \sin \frac{2x+2h-2x}{2}}{h}$$

$$2 \lim_{h \rightarrow 0} \cos \frac{4x+2h}{2} \lim_{h \rightarrow 0} \frac{\sin \frac{2h}{2}}{h}$$

$$2 \cos \frac{4x}{2} \cdot 1$$

$$\frac{d}{dx} \sin 2x = 2 \cos 2x$$

[ Practice:  $\sin x, \cos x$  ]

(ii)  $f(x) = \tan 2x$

$$f(x+h) = \tan 2(x+h) = \tan(2x+2h)$$

First principle

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \rightarrow 0} \frac{\tan(2x+2h) - \tan 2x}{h}$$

$$\lim_{h \rightarrow 0} \frac{\frac{\sin(2x+2h)}{\cos(2x+2h)} - \frac{\sin 2x}{\cos 2x}}{h}$$

$$\lim_{h \rightarrow 0} \frac{\frac{\sin(2x+2h)\cos 2x - \cos(2x+2h)\sin 2x}{\cos(2x+2h)\cos 2x}}{h}$$

$$\lim_{h \rightarrow 0} \frac{\sin[2x + 2h] - \sin 2x}{h \cos(2x + 2h) \cos 2x}$$

$$\lim_{h \rightarrow 0} \frac{\sin 2h}{2h} \cdot 2 \lim_{h \rightarrow 0} \frac{1}{\cos(2x + 2h) \cos 2x}$$

$$1 \cdot 2 \frac{1}{\cos^2 2x} = 2 \sec^2 2x$$

(iii)  $f(x) = ax^2 + bx + c$

First principle

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \rightarrow 0} \frac{[a(x+h)^2 + b(x+h) + c] - (ax^2 + bx + c)}{h}$$

$$\lim_{h \rightarrow 0} \frac{a(x^2 + 2hx + h^2) + b(x+h) - (ax^2 + bx + c)}{h}$$

$$\lim_{h \rightarrow 0} \frac{ax^2 + 2hax + ah^2 + bx + bh + c - ax^2 - bx - c}{h}$$

$$\lim_{h \rightarrow 0} \frac{h(2ax + b + ah)}{h} = \frac{d}{dx} f(x) = 2ax + b$$

2.  $x^3 + y^3 - 3axy = 0$  then find  $\frac{dy}{dx}$

A. Differentiate both sides with respect to 'x'

$$\frac{dy}{dx} x^3 + y^3 - 3axy = 0$$

$$3x^2 + 3y^2 \frac{dy}{dx} - 3a \left( x \frac{dy}{dx} + y \frac{d}{dx} x \right) = 0$$

$$x^2 + y^2 \frac{dy}{dx} - ax \frac{dy}{dx} - ay = 0$$

$$(y^2 - ax) \frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

3.  $y = e^{a \sin^{-1} x}$  then show that  $\frac{dy}{dx} = \frac{ay}{\sqrt{1-y^2}}$

A.  $y = e^{a \sin^{-1} x} \dots\dots(1)$

Differentiate both sides with respect to 'x'

$$\frac{dy}{dx} = \frac{d}{dx} e^{a \sin^{-1} x}$$

$$e^{a \sin^{-1} x} \frac{d}{dx} a \sin^{-1} x$$

$$\frac{dy}{dx} = e^{a \sin^{-1} x} \frac{a}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{ya}{\sqrt{1-x^2}} \text{ from (1)}$$

4. Find the derivative of  $\tan^{-1} \frac{a-x}{1+ax}$

A. put  $a = \tan A$   $A = \tan^{-1} a$

$$\tan B = B = \tan^{-1} c$$

$$\frac{d}{dx} \tan^{-1} \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\frac{d}{dx} \tan^{-1} \tan(A - B)$$

$$\frac{d}{dx} A - B$$

$$\frac{d}{dx} \tan^{-1} a - \tan^{-1} x$$

$$0 - \frac{1}{1-x^2}$$

$$-\frac{1}{1-x^2}$$

5.  $x^y = e^{x \cdot y}$  then show that  $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

A.  $x^y = e^{x \cdot y}$

take log both sides

$$\log_e x^y = \log_e e^{x \cdot y}$$

$$y \log x = (x - y) \log_e e$$

$$x - y = y \log x$$

$$x = y(1 + \log x)$$

$$y = \frac{x}{(1 + \log x)}$$

Differentiate both sides with respect to 'x'

$$\frac{dy}{dx} = \frac{d}{dx} \frac{x}{1 + \log x}$$

$$\frac{(1 + \log x) \cdot 1 - x \cdot \frac{1}{x}}{(1 + \log x)^2}$$

$$\frac{dy}{dx} = \frac{1 - \log x}{(1 + \log x)^2}$$

6.  $\sin y = x \sin(a + y)$  then show that  $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$

A.  $x = \frac{\sin y}{\sin(a + y)}$

Differentiate both sides with respect to 'y'

$$\frac{dx}{dy} = \frac{\sin(a + y) \cos y - \sin y \cos(a + y)}{\sin^2(a + y)}$$

$$\frac{dx}{dy} = \frac{\sin a - y - y}{\sin^2(a + y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$$

### Long Answer Questions (8 Marks)

1.  $f(x) = \frac{x \cos x}{\sqrt{1 + x^2}}$ , then find  $f'(x)$ .

A. Differentiate both sides with respect to 'x'

$$\frac{d}{dx} f(x) = \frac{d}{dx} \frac{x \cos x}{\sqrt{1 + x^2}}$$

$$f'(x) = \frac{\sqrt{1 + x^2} \frac{d}{dx} [x \cos x] - x \cos x \frac{d}{dx} \sqrt{1 + x^2}}{\sqrt{1 + x^2}^2}$$

$$\frac{\sqrt{1-x^2} \cdot x \frac{d}{dx} \cos x + \cos x \frac{d}{dx} x - x \cos x \frac{1}{2\sqrt{1-x^2}} \frac{d}{dx} (1-x^2)}{1-x^2}$$

$$\frac{\sqrt{1-x^2} \cdot x(-\sin x) + \cos x \cdot x - \frac{x \cos x}{2\sqrt{1-x^2}} (0-2x)}{1-x^2}$$

$$\frac{(1-x^2) (-x \sin x + \cos x) - x^2 \cos x}{\sqrt{1-x^2} (1-x^2)}$$

$$\frac{x(1-x^2) \sin x + \cos x - x^2 \cos x - x^2 \cos x}{(1-x^2)^{3/2}}$$

$$(1-x^2)^{3/2} [\cos x - x(1-x^2) \sin x]$$

**Practice Problem:** Find the value of  $\frac{d}{dx} \frac{x(1-x^2)}{\sqrt{1-x^2}}$

2.  $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ , then show that  $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

A. Let  $x = \sin A$ ,  $y = \sin B$ ,  $A = \sin^{-1} x$ ,  $B = \sin^{-1} y$

$$\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$$

Put  $x = \sin A$ ,  $y = \sin B$

$$\sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B} = a(\sin A - \sin B)$$

$$\cos A + \cos B = a(\sin A - \sin B)$$

$$2 \cos \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$$

$$2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos \frac{A+B}{2} = a \sin \frac{A-B}{2}$$

$$\tan \frac{A+B}{2} = \frac{1}{a}; \frac{1}{2}; \frac{1}{2} = \tan^{-1} \frac{1}{a}$$

$$2 \tan^{-1} \frac{1}{a};$$

$$\sin^{-1} y = \sin^{-1} x + 2 \tan^{-1} \frac{1}{a}$$

Differentiating w.r.to x

$$\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cdot \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

3. Find the derivation of  $\log \frac{x^2 + x + 2}{x^2 - x + 2}$ .

A.  $\frac{d}{dx} \log \frac{x^2 + x + 2}{x^2 - x + 2}$

$$\frac{d}{dx} \log(x^2 + x + 2) - \log(x^2 - x + 2)$$

$$\frac{1}{x^2 + x + 2} \frac{d}{dx}(x^2 + x + 2) - \frac{1}{x^2 - x + 2} \frac{d}{dx}(x^2 - x + 2)$$

$$\frac{2x + 1}{x^2 + x + 2} - \frac{2x - 1}{x^2 - x + 2}$$

$$\frac{(x^2 - x + 2)(2x + 1) - (2x - 1)(x^2 + x + 2)}{[(x^2 - 2) - x][(x^2 - 2) + x]}$$

$$\frac{4x^2 - 2x^2 - 4}{x^4 - 4 - 4x^2 - x^2}$$

$$\frac{4 - 2x^2}{x^4 - 3x^2 - 4}$$

4.  $y = \frac{x^3 \sqrt{2+3x}}{(2+x)(1-x)}$  then find  $\frac{dy}{dx}$

A. Take 'log' both sides

$$\log y = \log \frac{x^3 \sqrt{2+3x}}{(2+x)(1-x)}$$

$$\log x^3 + \log \sqrt{2+3x} - \log(2+x) - \log(1-x)$$

$$\log y = 3 \log x + \frac{1}{2} \log(2+3x) - \log(2+x) - \log(1-x)$$

Differentiate both sides with respect to 'x'

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{x} + \frac{3}{2(2+3x)} - \frac{1}{2+x} - \frac{1}{1-x}$$

$$\frac{dy}{dx} = y \left[ \frac{3}{x} + \frac{3}{2(2+3x)} - \frac{1}{2+x} - \frac{1}{1-x} \right]$$

5.  $y = x\sqrt{a^2 + x^2} + a^2 \log(x + \sqrt{a^2 + x^2})$  then show that  $\frac{dy}{dx} = 2\sqrt{a^2 + x^2}$

A. Differentiate both sides with respect to 'x'

$$\begin{aligned} \frac{dy}{dx} &= x \frac{1}{2\sqrt{a^2 + x^2}} \cdot 2x + \sqrt{a^2 + x^2} + \frac{a^2 \cdot 1}{x + \sqrt{a^2 + x^2}} \cdot \frac{1}{\sqrt{a^2 + x^2}} \cdot 2x \\ &= \frac{x^2}{\sqrt{a^2 + x^2}} + \frac{a^2 + \sqrt{a^2 + x^2} \cdot x}{\sqrt{a^2 + x^2} \cdot x \sqrt{a^2 + x^2}} \\ &= \frac{x^2 + a^2}{\sqrt{a^2 + x^2}} + \sqrt{a^2 + x^2} \\ &= \frac{\sqrt{a^2 + x^2} \cdot \sqrt{a^2 + x^2}}{\sqrt{a^2 + x^2}} + \sqrt{a^2 + x^2} \\ &= 2\sqrt{a^2 + x^2} \end{aligned}$$

6. Find the derivations of following functions.

A. (i)  $\sec \sqrt{\tan x}$

$$\begin{aligned} \frac{d}{dx} \sec \sqrt{\tan x} &= \sec \sqrt{\tan x} \tan \sqrt{\tan x} \frac{d}{dx} \sqrt{\tan x} \\ &= \sec \sqrt{\tan x} \tan \sqrt{\tan x} \frac{1}{2\sqrt{\tan x}} \frac{d}{dx} \tan x \\ &= \frac{\sec^2 x}{2\sqrt{\tan x}} \sec \sqrt{\tan x} \tan x \sqrt{\tan x} \end{aligned}$$

(ii)  $\frac{1 - \cos 2x}{1 + \cos 2x}$

$$\begin{aligned} \frac{d}{dx} \frac{1 - \cos 2x}{1 + \cos 2x} &= \frac{1 - \cos 2x}{1 + \cos 2x} \cdot 2 \sin^2 x + \frac{1 - \cos 2x}{1 + \cos 2x} \cdot 2 \cos^2 x \\ &= \frac{d}{dx} \frac{2 \sin^2 x}{2 \cos^2 x} \\ &= \frac{d}{dx} (\tan x)^2 \\ &= 2(\tan x)^2 \cdot \frac{d}{dx} \tan x \\ &= 2 \tan x \sec^2 x \end{aligned}$$

\*\*\*\*\*



## Errors and approximations

### Key concepts

A small change in  $x$  is  $\Delta x$

If  $x$  is changed as  $x + \Delta x$ , then change in  $y$  is  $\Delta y = f(x + \Delta x) - f(x)$

Differential in  $y$ ,  $dy = f'(x) \cdot \Delta x$

Relative error in  $y = \frac{\Delta y}{y}$

Percentage error in  $y = \frac{\Delta y}{y} \times 100$

### Short Answer Questions (4 Marks)

1. Find  $dy$ ,  $y$  for the following  $y = x^2 + 3x + 6$ ,  $x = 10$ ,  $\Delta x = 0.01$

**Sol:**  $\Delta y = f(x + \Delta x) - f(x)$

$$\Delta y = f(10 + 0.01) - f(10)$$

$$\Delta y = f(10.01) - f(10)$$

$$= (10.01)^2 + 3 \times 10.01 + 6 - (10^2 + 3 \times 10 + 6)$$

$$= 100.2001 + 30.03 + 6 - 100 - 30 - 6$$

$$= 130.2301 - 130$$

$$= 0.2301$$

$$dy = f'(x) \cdot \Delta x$$

$$(2x + 3) \cdot \Delta x$$

$$(2 \times 10 + 3) \cdot 0.01$$

$$= 23 \times 0.01$$

$$= 0.23$$

2. Find  $dy, y$  for the following function  $y = \cos x$  when  $x = 60^\circ, x = 61^\circ$

Sol:  $y = \cos x, x = 60^\circ$

$$y = f(x + \Delta x) - f(x)$$

$$y = f(60^\circ + 1^\circ) - f(60^\circ)$$

$$= f(61^\circ) - f(60^\circ)$$

$$= \cos 61^\circ - \cos 60^\circ$$

$$= 0.4848 - 0.5 = -0.0152$$

$$y = f'(x) \Delta x$$

$$= \sin x \cdot \Delta x$$

$$= -\sin 60^\circ \times 1^\circ$$

$$= -0.866 \times 0.0174 \quad (1^\circ = 0.0174 \text{ radians})$$

$$= -0.0150$$

3. Find  $dy, y$  for the following function  $y = e^x$  at  $x = 5, x = 5.02$

Sol:  $y = e^x, x = 5$

$$y = f(x + \Delta x) - f(x)$$

$$= f(5.02) - f(5)$$

$$= e^{5.02} - e^5$$

$$= e^{5.02} - e^5 + 0.02$$

$$y = f'(x) \Delta x$$

$$= (e^x - 1) \cdot \Delta x$$

$$= (e^5 - 1)(0.02)$$

4. Find  $dy, y$  for the following function  $y = 5x^2 + 6x + 6$  at  $x = 2, x = 2.001$

Sol:  $y = 5x^2 + 6x + 6, x = 2$

$$y = f(x + \Delta x) - f(x)$$

$$= f(2.001) - f(2)$$

$$= 5(2.001)^2 + 6 \times 2.001 + 6 - (5 \times 2^2 + 6 \times 2 + 6)$$

$$= 5(4.004001) + 12.006 + 6 - 20 - 12 - 6$$

$$= 0.026005$$

$$y = f'(x) \Delta x$$

$$= (10x + 6) \cdot \Delta x$$

$$= (10 \times 2 + 6)(0.001)$$

$$= 26 \times 0.001 = 0.026$$

5. If the increase in the side of a square is 2%, find the change in the area of the square.

Sol: Let side of the square  $x$

Area of the square  $A = x^2$

$$\text{given } \frac{x}{x} = 100 + 2 \dots\dots(1)$$

$$\frac{A}{dx} = 2x \quad \frac{A}{dx} = \frac{2x}{x^2}$$

$$\frac{A}{dx} = 100 + 2 \quad \frac{x}{x} = 100$$

$$= 2 \times 2 \text{ from (1)}$$

$$= 4$$

6. If the increase in the side of the square is 4% find the change in the area of the square.

Sol: Let side of the square  $x$

Area of the square  $A = x^2$

$$\frac{x}{x} = 100 + 4 \text{ (given)}$$

$$A = x^2 \quad \frac{A}{dx} = 2x$$

$$\frac{A}{dx} = 100 + 4 \quad x \quad \frac{2x}{x^2} = 100 + 4$$

$$= 2 \times \frac{x}{x} = 100 + 4$$

$$= 2 \times 4 \text{ from (1)}$$

$$= 8$$

7. The radius of the sphere is measured is 14 c.m., Later it was found that there is an error 0.02cm in measuring the radius, find the approximate error in surface area of the sphere.

Sol: Let radius of the sphere  $r$

$$\text{Surface Area } A = 4\pi r^2 \quad \frac{dA}{dr} = 4 \times 2r$$

$$dA = \frac{dA}{dr} \times r$$

$$4 \quad 2r \quad r$$

$$8 \quad \frac{22}{7} \quad 14 \quad 0.02$$

$$= 7.04 \text{ sq.cm.}$$

## Geometrical interpretation of the derivative

### Key concepts

Slope of the tangent at point  $(x, y)$  on the curve  $y = f(x)$  is  $\frac{dy}{dx}$

Equation of tangent at point  $(a, b)$  on the curve  $y = f(x)$  is  $y - b = m(x - a) \quad \because m = \frac{dy}{dx}$

Slope of the normal =  $\frac{1}{m}$

Equation of the normal is  $y - b = \frac{1}{m}(x - a)$

1. Find the slope of the tangents to the curve  $y = 3x^2 - x^3$ , where it meets the X-axis.

**Sol:**  $y = 3x^2 - x^3$  ——— (1)

Equation of  $x$ -axis is  $y = 0$

from and

$$x^2 - x^3 = 0 \quad (\because y = 0)$$

$$x^2(3 - x) = 0$$

$$x^2 = 0 \quad \text{or} \quad 3 - x = 0$$

$$x = 0 \quad \text{or} \quad x = 3$$

The given curve intersects  $x$ -axis at the points

$$\begin{aligned} \text{Slope of tangent at } (0, 0), m &= \frac{dy}{dx} \Big|_{0,0} \\ &= 6x - 3x^2 \\ &= 6 \cdot 0 - 3 \cdot 0 \\ &= 0 \end{aligned}$$

Equation of tangent at  $(0, 0)$  is

$$y - 0 = 0(x - 0)$$

$$y = 0$$

$$\begin{aligned} \text{Slope of tangent at } (3, 0), m &= \frac{dy}{dx} \Big|_{3,0} \\ &= 6x - 3x^2 \\ &= 6 \cdot 3 - 3 \cdot 3^2 \\ &= 18 - 27 = -9 \end{aligned}$$

Equation of tangent at (3, 0) is

$$\begin{aligned} y - 0 &= -9(x - 3) \\ y &= -9x + 27 \\ 9x + y - 27 &= 0 \end{aligned}$$

**2. Show that the area of the triangle formed by the tangent at any point on the curve  $xy = c$  (C > 0) with the co-ordinate axes is constant.**

**Sol:** Let P  $(x_1, y_1)$  be the point on the curve  $xy = c$  and  $x_1 > 0, y_1 > 0$

$$y = \frac{c}{x}$$

Defferentiating both sides w.r.t. 'x'

$$\frac{dy}{dx} = -\frac{c}{x^2}$$

Slope of tangent,  $m = -\frac{c}{x_1^2}$  (at P  $(x_1, y_1)$ )

Equation of the tangent at P  $(x_1, y_1)$  is

$$y - y_1 = -\frac{c}{x_1^2} (x - x_1)$$

$$y x_1^2 - y_1 x_1^2 = -cx + cx_1$$

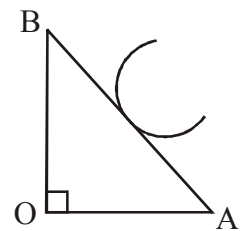
$$cx - yx_1^2 = cx_1 - y_1 x_1^2$$

$$cx - yx_1^2 = cx_1 - y_1 x_1 x_1$$

$$cx - yx_1^2 = cx_1 - cx_1 \quad \because x_1 y_1 = c$$

$$cx - yx_1^2 = 2cx_1$$

$$cx - yx_1^2 - 2cx_1 = 0$$



$$cx = x_1^2 y = 2cx_1$$

$$\frac{cx}{2cx_1} = \frac{x_1^2 y}{2cx_1} = 1$$

$$\frac{x}{2x_1} = \frac{y}{\frac{2c}{x_1}} = 1$$

$$\text{area of the triangle} = \frac{1}{2} \cdot 2x_1 \cdot \frac{2c}{x_1} = 2c$$

**3. Find the slope of the tangent to the curve  $y = 3x^4 - 4x$  at  $x = 4$**

**Sol:**  $y = 3x^4 - 4x$

Differentiate both sides w.r.t. 'x'

$$\frac{dy}{dx} = 3 \cdot 4x^3 - 4$$

$$\begin{aligned} \text{slope of the tangent at } x = 4 \quad m &= 12 \cdot 4^3 - 4 \\ &= 12 \cdot 64 - 4 \\ &= 768 - 4 = 764 \end{aligned}$$

**4(i). Find the slope of the tangent to the curve  $y = x^3 - 3x + 2$  at the point whose  $x$  coordinate is 3.**

**Sol:**  $y = x^3 - 3x + 2$

Differentiate both sides w.r.t. 'x'

$$\frac{dy}{dx} = 3x^2 - 3$$

$$\begin{aligned} \text{slope of the tangent at } x = 3, \quad m &= 3 \cdot 3^2 - 3 \\ &= 27 - 3 = 24 \end{aligned}$$

**4(ii). Find the slope of the normal to the curve  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$  at  $\theta = \frac{\pi}{4}$**

**Sol:**  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$

Differentiating with respect to

$$\frac{dx}{d\theta} = a \cdot 3 \cos^2 \theta \cdot (-\sin \theta)$$

$$\frac{dy}{dx} = a - 3\sin^2 \theta \cos \theta$$

slope of the tangent,  $m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta \sin \theta} = \tan \theta = \tan \frac{\pi}{4}$

$$m = -1$$

slope of the normal =  $\frac{1}{m} = \frac{1}{-1} = -1$

5. Find the slope of the normal to the curve  $x = 1 - a \sin \theta$ ,  $y = b \cos^3 \theta$  at  $\theta = \frac{\pi}{4}$

Sol:  $x = 1 - a \sin \theta$ ,  $y = b \cos^3 \theta$

differentiating with respect to  $\theta$

$$\frac{dx}{d\theta} = -a \cos \theta \quad \frac{dy}{d\theta} = 2b \cos \theta \sin \theta$$

slope of the tangent at  $\theta = \frac{\pi}{2}$   $m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2b \cos \theta \sin \theta}{-a \cos \theta}$

$$= \frac{2b}{-a} \sin \frac{\pi}{2}$$

$$m = \frac{2b}{-a}$$

slope of the normal at  $\theta = \frac{\pi}{2}$   $= \frac{1}{m}$

$$= \frac{1}{2b/-a}$$

$$= \frac{a}{2b}$$

6. Find the equation of tangent and normal to the curve  $y = x^2 - 4x + 2$  at point (4, 2)

Sol:  $y = x^2 - 4x + 2$

differentiating with respect to  $x$

$$\frac{dy}{dx} = 2x - 4$$

slope of the tangent at (4, 2),  $m = 2 \cdot 4 - 4 = 4$

equation of the tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 4(x - 4)$$

$$y - 2 = 4x - 16$$

$$4x - y - 16 + 2 = 0$$

$$4x - y - 14 = 0$$

slope of the normal  $\frac{1}{m} = \frac{1}{4}$

equation of the normal is,  $y - 2 = \frac{1}{4}(x - 4)$

$$4y - 8 = x - 4$$

$$x + 4y - 8 - 4 = 0$$

$$x + 4y - 12 = 0$$

**7. Find the equation of tangent and normal to the curve  $y = x^3 + 4x^2$  at  $(-1, 3)$**

**Sol:**  $y = x^3 + 4x^2$

differentiating with respect to  $x$

$$\frac{dy}{dx} = 3x^2 + 8x$$

slope of the tangent at  $m = 3(-1)^2 + 8(-1)$   
 $= 3 - 8$   
 $= -5$

equation of the tangent is,  $y - 3 = -5(x + 1)$

$$y - 3 = -5x - 5$$

$$5x + y + 2 = 0$$

slope of the normal =  $\frac{1}{m}$

$$= \frac{1}{-5} = -\frac{1}{5}$$

equation of the normal  $y - 3 = -\frac{1}{5}(x + 1)$

$$5y - 15 = x + 1$$

$$x - 5y + 16 = 0$$



8. Find the tangent and normal to the curve  $y = 2e^{\frac{x}{3}}$  at the point where the curve meets y-axis.

Sol: Equation of y-axis  $x = 0$

$$y = 2e^{\frac{x}{3}}, \text{ the curve meets y-axis at } [\because y = 2e^{\frac{x}{3}} = 2]$$

$$y = 2e^{\frac{x}{3}}$$

differentiating with respect to x

$$\begin{aligned} \frac{dy}{dx} &= 2e^{x/3} \cdot \frac{1}{3} \\ &= \frac{2}{3} e^{x/3} \end{aligned}$$

$$\text{slope of the tangent at } x = 0 \quad m = \frac{2}{3} e^{0/3} = \frac{2}{3}$$

equation of the tangent at  $x = 0$  is

$$y - 2 = \frac{2}{3} (x - 0)$$

$$3y - 6 = 2x$$

$$2x - 3y + 6 = 0$$

$$\text{slope of the normal } = \frac{1}{m} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$$\text{equation of the normal } y - 2 = \frac{3}{2} (x - 0)$$

$$2y - 4 = 3x$$

$$3x - 2y + 4 = 0$$

9. If the tangent at any point on the curve  $x^{2/3} + y^{2/3} = a^{2/3}$  intersects the co-ordinate axes in A and B then show that the length of AB is constant.

Sol: P  $(x_1, y_1)$  be the any point the on the curve  $x^{2/3} + y^{2/3} = a^{2/3}$

$$x_1^{2/3} + y_1^{2/3} = a^{2/3} \quad \dots (1)$$

$$x^{2/3} + y^{2/3} = a^{2/3}$$

differentiating both sides with respect to 'x'

$$\frac{2}{3} x^{-\frac{1}{3}} - \frac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y^{1/3}}{x^{1/3}}$$

slope of the tangent at  $P(x_1, y_1)$

$$m = \frac{y_1^{1/3}}{x_1^{1/3}}$$

equation of the tangent at  $P(x_1, y_1)$

$$y - y_1 = \frac{y_1^{1/3}}{x_1^{1/3}} (x - x_1)$$

$$y x_1^{1/3} - y_1 x_1^{1/3} = x y_1^{1/3} - y_1^{1/3} x_1$$

$$x y_1^{1/3} - y x_1^{1/3} = x_1 y_1^{1/3} - x_1^{1/3} y_1$$

Dividing by  $y_1^{1/3} x_1^{1/3}$ , we get

$$\frac{x y_1^{1/3}}{y_1^{1/3} x_1^{1/3}} - \frac{y x_1^{1/3}}{y_1^{1/3} x_1^{1/3}} = \frac{x_1 y_1^{1/3}}{y_1^{1/3} x_1^{1/3}} - \frac{x_1^{1/3} y_1}{y_1^{1/3} x_1^{1/3}}$$

$$\frac{x}{x_1^{1/3}} - \frac{y}{y_1^{1/3}} = x_1^{2/3} - y_1^{2/3}$$

$$\frac{x}{x_1^{1/3}} - \frac{y}{y_1^{1/3}} = a^{2/3} \quad (\text{from eqn-1})$$

above equation intersects x-axis at  $A (a^{2/3} x_1^{1/3}, 0)$ , y-axis at  $B (0, a^{2/3} y_1^{1/3})$

$$\begin{aligned} AB &= \sqrt{a^{2/3} x_1^{2/3} + a^{2/3} y_1^{2/3}} \\ &= \sqrt{a^{2/3} (x_1^{2/3} + y_1^{2/3})} \\ &= \sqrt{a^{4/3} a^{2/3}} \quad (\text{from eqn-1}) \\ &= \sqrt{a^{6/3}} \\ &= \sqrt{a^2} \\ &= a = \text{constant} \end{aligned}$$

**Angle between two curves and condition for orthogonality of curves**

Angle between tangents drawn at intersecting points of two curves  $C_1$  and  $C_2$  is angle between the curves  $C_1$  and  $C_2$

If slopes of tangents drawn at intersecting points of curves is  $m_1$  and  $m_2$  and angle between

the curves is ' $\theta$ ' then  $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

If  $m_1 = m_2$ , the curves have common tangents, and touch each other.

If  $m_1 \cdot m_2 = -1$  the curves intersect orthogonally.

**1. Show that the curves  $y^2 = 4(x+1)$ ,  $y^2 = 36(9-x)$  intersect orthogonally.**

**Sol:**  $y^2 = 4(x+1)$ ,  $y^2 = 36(9-x)$

$$4(x+1) = 36(9-x)$$

$$x + 1 = 9(9-x)$$

$$x + 1 = 81 - 9x$$

$$x + 9x = 81 - 1$$

$$10x = 80$$

$$x = 8$$

$$y^2 = 4(x + 1)$$

$$y^2 = 4(8 + 1)$$

$$y^2 = 4 \cdot 9 = 36$$

$$y = \pm 6$$

intersecting points of given curves are

slope of tangent at 'p' to the curve  $y^2 = 4(x + 1)$

$$2y \frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y} \text{ (first curve)}$$

$$y^2 = 36(9-x) \quad 2y \frac{dy}{dx} = -36$$

$$\frac{dy}{dx} = \frac{-36}{2y} = \frac{-18}{y} \text{ (second curve)}$$

slope of the curve  $y^2 = 4(x + 1)$  at  $(2, 6)$   $m_1 = \frac{2}{6} = \frac{1}{3}$

slope of the curve  $y^2 = 36(9 - x)$  at  $(2, 6)$   $m_2 = -\frac{18}{6} = -3$

$$m_1 \cdot m_2 = \frac{1}{3} \cdot (-3) = -1$$

The given curves intersect orthogonally. Similarly we can prove at point  $Q(8, -6)$

**2. Show that the condition for the orthogonality of curves  $ax^2 + bx = 1$ ,  $a_1x^2 + b_1y^2 = 1$  is**

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$$

**Sol:** Let intersecting point of given curves  $ax^2 + bx = 1$ ,  $a_1x^2 + b_1y^2 = 1$  be

$P(x_1, y_1)$

$$ax_1^2 + bx_1 - 1 = 0, \quad a_1x_1^2 + b_1y_1^2 - 1 = 0$$

$$\begin{array}{r} x_1^2 \quad y_1^2 \quad 1 \\ b \quad 1 \quad a \quad b \\ b_1 \quad 1 \quad a_1 \quad b_1 \end{array}$$

$$\frac{x_1^2}{b} - \frac{y_1^2}{b_1} = \frac{y_1^2}{a_1} - \frac{1}{a} = \frac{1}{ab_1} - \frac{1}{a_1b} \quad \dots (1)$$

slope of the tangent to the curve  $ax^2 + by^2 = 1$  at  $P(x_1, y_1)$

$$2ax + 2by^2 \cdot y^1 = 0$$

$$y_1 = \frac{2ax}{2by} = \frac{ax}{by}$$

$$m_1 = \frac{ax_1}{by_1}$$

slope of the tangent to the curve  $a_1x^2 + b_1y^2 = 1$  at  $P(x_1, y_1)$

$$m_1 = \frac{a_1x_1}{b_1y_1}$$

condition for orthogonality is  $m_1 \cdot m_2 = -1$

$$\frac{ax_1}{by_1} \cdot \frac{a_1x_1}{b_1y_1} = -1$$

$$\frac{aa_1}{bb_1} \frac{x_1^2}{y_1^2} = 1$$

$$\frac{x_1^2}{y_1^2} = \frac{bb_1}{aa_1}$$

$$\frac{b_1}{a} \frac{b}{a_1} = \frac{bb_1}{aa_1}$$

from eqn---(1)

$$b_1 \cdot aa_1 - b \cdot aa_1 = -abb_1 + ba_1b_1$$

$$a a_1 b_1 - ba_1 b_1 = aba_1 - abb_1$$

$$a_1 b_1 (a-b) = ab (a_1-b_1)$$

$$\frac{a}{ab} \frac{b}{a_1 b_1} = \frac{a_1}{a_1 b_1} \frac{b_1}{a_1 b_1}$$

$$\frac{a}{ab} \frac{b}{ab} = \frac{a_1}{a_1 b_1} \frac{b_1}{a_1 b_1}$$

$$\frac{1}{b} \frac{1}{a} = \frac{1}{b_1} \frac{1}{a_1}$$

$$\frac{1}{a} \frac{1}{b} = \frac{1}{a_1} \frac{1}{b_1}$$

**3. Find the angle between the curves  $x + y + 2 = 0$ ,  $x^2 + y^2 - 10y = 0$**

**Sol:**  $x + y + 2 = 0$

$$x = -(y + 2)$$

$$x^2 + y^2 - 10y = 0$$

$$(-(y + 2))^2 + y^2 - 10y = 0$$

$$y^2 + 4y + 4 + y^2 - 10y = 0$$

$$2y^2 - 6y + 4 = 0$$

$$y^2 - 3y + 2 = 0$$

$$(y - 1)(y - 2) = 0$$

$$y = +1 \text{ or } y = 2$$

$$\text{at } y = 1, x = -(1 + 2) = -3$$

$$\text{at } y = 2, x = -(2 + 2) = -4$$

Intersecting points are  $(-3, 1)$ ,  $(-4, 2)$

slope of the tangent at  $(-3, 1)$  to the curve  $x + y + 2 = 0$

$$1 + y^1 = 0$$

$$y^1 = -1$$

$$m_1 = -1$$

$$x^2 + y^2 - 10y = 0$$

differentiating both sides with respect to 'x'

$$2x + 2y \cdot y^1 - 10 \cdot y^1 = 0.$$

$$y^1 = \frac{2x}{2y - 10}$$

$$m_2 = \frac{2}{2} \cdot \frac{3}{1} \cdot \frac{3}{10}$$

$$= \frac{6}{2} \cdot \frac{6}{8} \cdot \frac{3}{4}$$

If angle between two curves is  $\theta$ , then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{1 - 3/4}{1 + 1 \cdot 3/4} \right|$$

$$= \left| \frac{\frac{4}{4} - \frac{3}{4}}{\frac{4}{4} + \frac{3}{4}} \right|$$

$$= \left| \frac{1}{7} \right|$$

$$= \frac{1}{7}$$

$$= \tan^{-1} \frac{1}{7}$$

**4. Find the angle between the curves  $y^2 = 4x$ ,  $x^2 + y^2 = 5$**

**Sol:**  $y^2 = 4x$ ,  $x^2 + y^2 = 5$

$$x^2 + 4x - 5 = 0$$

$$(x + 5)(x - 1) = 0$$

$$x = -5 \text{ or } 1$$

$$x = 1 \quad y^2 = 4 \quad 1$$

$$y^2 = 4$$

$$y = \pm 2$$

intersecting points of two curves

$$y^2 = 4x$$

differentiating both sides with respect to 'x'

$$2y \cdot y' = 4$$

$$y' = \frac{4}{2y} = \frac{2}{y}$$

slope of the first curve at  $\left(\frac{2}{2}, 1\right) (m_1)$

$$x^2 + y^2 = 5$$

differentiating both sides with respect to 'x'

$$2x + 2yy' = 0 \quad y' = -\frac{x}{y}$$

slope of the second curve at

$$m_2 = -\frac{1}{2}$$

If angle between two curves is  $\theta$  then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\tan \theta = \left| \frac{1 - \left(-\frac{1}{2}\right)}{1 + 1 \cdot \left(-\frac{1}{2}\right)} \right|$$

$$\left| \frac{\frac{3}{2}}{\frac{1}{2}} \right| = 3$$

$$\tan \theta = 3$$

$$\theta = \tan^{-1}(3)$$

5. Find the angle between two curves  $x^2 = 2(y+1)$  ;  $y = \frac{8}{x^2} - 4$

**Sol:**  $x^2 = 2(y + 1), y = \frac{8}{x^2} - 4$

$$x^2 + 4 = \frac{8}{y}$$

$$x^2 = \frac{8}{y} - 4$$

$$\frac{8}{y} - 4 = 2(y + 1)$$

$$\frac{8}{y} = 2y + 2 + 4$$

$$\frac{8}{y} = 2y + 6$$

$$\frac{4}{y} = y + 3$$

$$4 = y^2 + 3y$$

$$y^2 + 3y - 4 = 0$$

$$(y + 4)(y - 1) = 0$$

$$y = -4 \text{ or } y = 1$$

$$x^2 = 2(1 + 1)$$

$$x^2 = 2 \cdot 2$$

$$x = \pm 2$$

intersecting points are  $(-2, 1)$  and  $(2, 1)$

$$x^2 = 2(y + 1)$$

differentiating both sides with respect to 'x'

$$2x = 2y^1$$

$$y^1 = x$$

slope of the curve at  $(2, 1)$  is  $m_1$

$$y = \frac{8}{x^2} - 4$$

differentiating both sides with respect to 'x'



$$y^1 = \frac{8}{x^2 - 4} 2x$$

slope of the second curve at  $x = 2$  to the curve  $y = \frac{8}{x^2 - 4}$

$$m_2 = \frac{16 \cdot 2}{2^2 - 4} = \frac{32}{0} = \frac{1}{2}$$

$$m_1 \cdot m_2 = 2 \cdot \frac{1}{2} = 1$$

the given curves intersect orthogonally.

6. Show that the curves  $6x^2 - 5x + 2y = 0$ ,  $4x^2 + 8y^2 = 3$  touch each other at points

$$\left(\frac{1}{2}, \frac{1}{2}\right)$$

**Sol:**  $6x^2 - 5x + 2y = 0$

differentiating both sides with respect to 'x'

$$12x - 5 + 2y^1 = 0$$

$$y^1 = 5 - 12x$$

$$y^1 = \frac{5 - 12x}{2}$$

slope of the tangent at  $\left(\frac{1}{2}, \frac{1}{2}\right)$  to the first curve is  $m_1 = \frac{5 - 12 \cdot \frac{1}{2}}{2} = \frac{5 - 6}{2} = -\frac{1}{2}$

$$4x^2 + 8y^2 = 3$$

differentiating both sides with respect to 'x'

$$8x + 16y \cdot y^1 = 0$$

$$y^1 = -\frac{8x}{16y} = -\frac{x}{2y}$$

slope of the tangent at  $\frac{1}{2}, \frac{1}{2}$  to the second curve is  $m_2 = \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2}$

$$m_1 = m_2$$

the given curves touch each other at  $\frac{1}{3}, \frac{1}{2}$

## Maxima and Minima

### Key concepts

#### 1. 1<sup>st</sup> derivative test :

Let  $f$  be a differential function on an interval  $D$ ,  $c \in D$ , and  $f$  is defined in some neighbourhood of  $c$ , suppose,  $c$  is a stationary point of  $f$  such that  $(c - \delta, c + \delta)$  does not contain any other stationary point for some  $\delta > 0$  then.

- (i)  $c$  is a point of local maximum, if  $f'(x)$  changes sign from positive to negative at  $x = c$
- (ii)  $c$  is a point of local minimum, if  $f'(x)$  changes sign from negative to positive at  $x = c$
- (iii)  $c$  is neither a point of local maximum nor a points of local minimum  $f'(x)$  does not change sign at  $x = c$

#### 2. 2<sup>nd</sup> derivative test :

- (i)  $x=c$  is a point of local maximum of ' $f$ ' if  $f'(c)=0$  and  $f''(c) < 0$ , and local maximum value of ' $f$ ' is  $f(c)$
- (ii)  $x=c$  is a point of local minimum of ' $f$ ' if  $f'(c) = 0$  and  $f''(c) > 0$  and local minimum value ' $f$ ' is  $f(c)$
- (iii) the test fails if  $f'(c) \neq 0$  and  $f''(c) = 0$

#### 1. Find the maximum area of the rectangle that can be formed with fixed perimeter 20.

**Sol:** Let  $x$  and  $y$  denote the length and the breadth of a rectangle respectively . Given that the perimeter of the rectangle is 20.

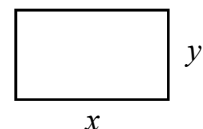
$$2(x + y) = 20 \quad \dots\dots (1)$$

$$x + y = 10 \quad \dots\dots\dots(2)$$

$$\text{area of the rectangle } A = x.y. \quad \dots\dots\dots (3)$$

$$A = x \cdot (10 - x) \quad \because x + y = 10$$

$$A = 10x - x^2 \quad \dots\dots\dots (4)$$



differentiating both sides with respect to 'x'

$$\frac{dA}{dx} = 10 - 2x$$

$$\frac{dA}{dx} = 0 \quad 10 - 2x = 0$$

$$2x = 10$$

$$x = 5$$

$x = 5$  is the stationary point

differentiating (5) with respect to 'x'

$$\frac{d^2A}{dx^2} = -2$$

$$\frac{d^2A}{dx^2} < 0$$

which is negative, therefore by second derivative test the area A is maximum at  $x = 5$  and hence  $y = 10 - 5 = 5$ , and the maximum area is  $A = x \cdot y = 5(5) = 25$

- 2. The profit function  $P(x)$  of a company selling  $x$  items per day is given by  $P(x) = (150 - x)x - 1600$ . Find the number of items that the company should sell to get maximum profit. Also find the maximum profit.**

**Sol:** 
$$P(x) = (150 - x)x - 1600$$

$$= 150x - x^2 - 1600$$

differentiating both sides with respect to 'x'

$$\frac{dP}{dx} = 150 - 2x$$

$$\frac{dP}{dx} = 0$$

$$150 - 2x = 0$$

$$2x = 150$$

$$x = 75$$

Again differentiating Eqn .... (1) with respect to 'x'

$$\frac{d^2P}{dx^2} = -2$$

$$\frac{d^2p}{dx^2} = -2$$

The profit  $P(x)$  is maximum for  $x = 75$

The company should sell 75 items a day to make maximum profit.

$$\begin{aligned} \text{the maximum profit } P(75) &= (150 - 75) \cdot 75 - 1600 \\ &= 75 \cdot 75 - 1600 \\ &= 5625 - 1600 \\ &= 4025. \end{aligned}$$

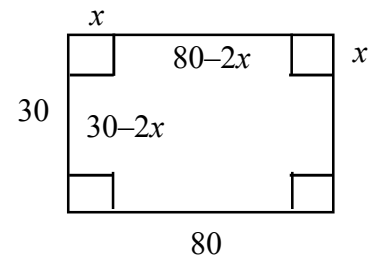
3. From a rectangular sheet of dimensions  $30\text{cm} \times 80\text{cm}$  four equal squares of side  $x$  cm are removed at the corners, and the sides are then turned up so as to form an open rectangular box. Find the value of  $x$ , so that the volume of the box is the greatest.

**Sol:** length of the box =  $80 - 2x$

breadth of the box =  $30 - 2x$

height of the box =  $x$

$$\begin{aligned} \text{volume } V &= (80 - 2x)(30 - 2x) \cdot x \\ &= (2400 - 160x - 60x + 4x^2) \cdot x \\ V &= 4x^3 - 220x^2 + 2400x \end{aligned}$$



differentiating both sides with respect to 'x'

$$\frac{dV}{dx} = 12x^2 - 440x + 2400$$

$$\frac{dV}{dx} = 0$$

$$12x^2 - 440x + 2400 = 0$$

$$3x^2 - 110x + 600 = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{110 \pm \sqrt{110^2 - 4 \cdot 3 \cdot 600}}{2 \cdot 3} \\ x &= \frac{110 \pm \sqrt{12100 - 7200}}{6} \end{aligned}$$

$$\begin{aligned}
 &= \frac{110 \sqrt{4900}}{6} \\
 &= \frac{110 \cdot 70}{6} \\
 &= \frac{110 \cdot 70}{6} = \frac{110 \cdot 70}{6} \\
 &= \frac{180}{6} = \frac{40}{6} \\
 &= \frac{20}{3}
 \end{aligned}$$

$$\begin{aligned}
 x = 30 \quad b = 30 - 2 \cdot 30 \\
 &= -30 \\
 b < 0 \\
 x &= 30 \\
 x &= \frac{20}{3}
 \end{aligned}$$

Again differentiating Eqn .... (1) with respect to 'x'

$$\frac{d^2v}{dx^2} = 24x - 440$$

$$\begin{aligned}
 \text{For } x = \frac{20}{3} \quad \frac{d^2v}{dx^2} &= 24 \cdot \frac{20}{3} - 440 \\
 &= 160 - 440 \\
 &= -280
 \end{aligned}$$

$$\frac{d^2v}{dx^2} < 0$$

Volume of box is maximum at  $x = \frac{20}{3}$

4. A window is in the shape of a rectangle surrounded by a semicircle. If the perimeter of the window is 20 ft. Find the maximum area.

Sol: The perimeter of window =  $22 + 2y = x = 20$

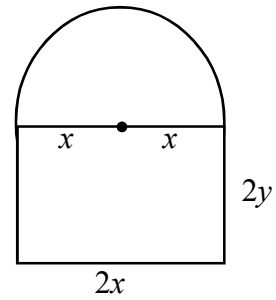
$$2y = 20 - (x + 2) \cdot x.$$

$$y = 10 - \frac{2}{2}x$$

$$\text{area} = 2xy - \frac{x^2}{2}$$

$$A = 20 - 2x - \frac{x^2}{2}$$

$$= 20 - 2x - \frac{x^2}{2}$$



differentiating both sides with respect to 'x'

$$\frac{dA}{dx} = 20 - 2 - 2x - \frac{2x}{2}$$

$$\frac{dA}{dx} = 0$$

$$20 - (2 + 2x) = 0$$

$$x(-2 - 4) = -20$$

$$x(-6) = -20$$

$$x(-4) = 20$$

$$x = \frac{20}{4}$$

Again differentiating Eqn .... (1) with respect to 'x'

$$\frac{d^2A}{dx^2} = - (2 + 2)$$

$$= -2 - 4$$

$$= -6$$

$$\frac{d^2A}{dx^2} < 0$$

area of the window maximum at  $x = \frac{20}{4}$

$$A = 2y - \frac{x^2}{2}$$

$$2y = 20 - (2)x$$

$$2y = 20 - 2 \cdot \frac{20}{4}$$

$$2y = \frac{20 - 80 + 20}{4} = \frac{40}{4}$$

$$A = 2xy - \frac{1}{2} x^2$$

$$= \frac{20}{4} \cdot \frac{40}{4} - \frac{1}{2} \cdot \left(\frac{20}{4}\right)^2$$

$$= \frac{1600}{2} - \frac{400}{4}$$

$$= \frac{2000 - 400}{4} = \frac{1600}{4} = 400$$

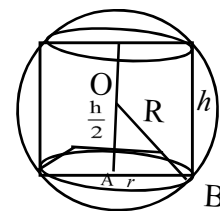
5. If the curved surface of right circular cylinder inscribed in a sphere of radius  $r$  is maximum, show that the height of the cylinder is  $\sqrt{2} r$ .

Sol: From  $\triangle OAB$

$$OA^2 + AB^2 = OB^2$$

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$

$$r^2 = R^2 - \frac{h^2}{4}$$



lateral surface area of cylinder =  $2 \pi r h$

$$= 2 \pi \sqrt{R^2 - \frac{h^2}{4}} \cdot h$$

$$A = \frac{2 \pi h}{2} \sqrt{4R^2 - h^2}$$

$$A = \pi h \sqrt{4R^2 - h^2}$$

differentiating both sides with respect to 'h'

$$\frac{dA}{dh} = 1 \cdot \sqrt{4R^2 + h^2} - \frac{h \cdot 1}{\sqrt{4R^2 + h^2}}$$

$$\frac{dA}{dh} = \frac{4R^2 + h^2 - h^2}{\sqrt{4R^2 + h^2}}$$

$$\frac{4R^2 - 2h^2}{\sqrt{4R^2 + h^2}}$$

$$\frac{dA}{dh} = 0$$

$$\frac{4R^2 - 2h^2}{\sqrt{4R^2 + h^2}} = 0 \quad 4R^2 - 2h^2 = 0$$

$$h = \sqrt{2} R.$$

Again differentiating Eqn .... (1) with respect to 'h'

$$\frac{d^2A}{dh^2} = 2 \frac{\sqrt{4R^2 + h^2} - 2h \frac{2R^2 + h^2}{2\sqrt{4R^2 + h^2}}}{4R^2 + h^2} = 2 \frac{\sqrt{4R^2 + h^2} - \frac{2R^2 + h^2}{\sqrt{4R^2 + h^2}}}{4R^2 + h^2}$$

$$= 2 \frac{8R^2h - 2h^3 - 2R^2h - h^3}{4R^2 + h^2} = 2 \frac{8R^2h - 2h^3 - 2R^2h - h^3}{4R^2 + h^2}$$

$$= 2 \frac{8R^2h - 2h^3 - 2R^2h - h^3}{4R^2 + h^2}$$

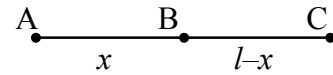
$$= \frac{4h}{\sqrt{4R^2 + h^2}} = 0 \quad \text{at } h = \sqrt{2}R \quad \frac{d^2A}{dh^2} < 0$$

Lateral surface area of cylinder is maximum when  $h = \sqrt{2}R$



6. A wire of length  $l$  is cut into two parts which are bent respectively in the form of a square and a circle. What are the lengths of the pieces of the wire respectively so that the sum of the areas is the least.

**Sol:** Let square is formed with the wire of length  $x$   
 Let circle is formed with wire of length is  $l-x$



$$\text{side of the square} = \frac{x}{4}$$

$$\text{area} = \frac{x^2}{4}$$

$$\text{circumfrance of the circle, } 2\pi r = l - x$$

$$r = \frac{l - x}{2\pi}$$

$$\text{area of the circle} = \pi r^2 = \frac{\pi (l - x)^2}{4}$$

$$\text{sum of the areas, } A = \frac{x^2}{4} + \frac{\pi (l - x)^2}{4}$$

differintating both sides with respect to 'x'

$$\frac{dA}{dx} = \frac{2x}{4} - \frac{2\pi (l - x)}{4} = 0 \quad (1)$$

$$\frac{dA}{dx} = 0$$

$$\frac{2x}{4} - \frac{2\pi (l - x)}{4} = 0 \quad \frac{x}{2} - \frac{\pi (l - x)}{2} = 0$$

$$\frac{x - \pi l + \pi x}{2} = 0 \quad x - \pi l + \pi x = 0$$

$$x + \pi x = \pi l$$

Again differintating Eqn .... (1) with respect to 'x'

$$\frac{d^2A}{dx^2} = \frac{1}{2} + \frac{\pi}{2}$$

$$\frac{1}{8} - \frac{1}{2} = 0$$

At  $x = \frac{4l}{4}$  we get minimum value of A

$$l - x = l - \frac{4l}{4} = \frac{l - 4l}{4} = -\frac{3l}{4}$$

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